Lectures on Entanglement in QFTs and Holography

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Abstract: These lecture notes involve an extensive review of the notions of entanglement in QFTs and their corresponding holographic duals in the context of AdS/CFT correspondence. These lectures were delivered in Ramakrishna Mission Vivekananda Educational and Research Institute (RKMVERI) Belur Math, Howrah, India

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1 Entanglement in Quantum Information Theory

To begin with let us try to motivate the question why is entanglement interesting and what exactly we mean by entanglement? Classically when we consider a bipartite system they can be completely uncorrelated if they are not interacting with each other or they can have some correlations if they are/were causally connected at some point in time. However, when we make a measurement on system A or system B which are casually disconnected we do not expect their measurements to influence each other. However, we will see in the first part of the lecture that quantum mechanics allows for much stronger correlations than any classical theory. Historically this was first discovered by John Bell in 1964. We start with a version of Bell's inequality known as CHSH inequality which might be slightly different from the way Bell himself considered it, nevertheless communicates the physics.

1.1 Bell's inequality and Local Realism

In order to understand the implication of Bell's inequality one needs to understand the concept of local realism. Locality is a principle in which two causally disconnected events can not influence each other. In other words suppose you have two physical systems which are so far apart that even light signals can not be sent in time then the measurements can not affect each other. Realism simply means that the measured values of a property of a physical system objectively were so even before the measurement was made. Theories which obey both of these two principles are called locally realistic theories.

John Bell worked made this mathematically precise by constructing certain inequalities which any locally realistic theory must obey. We will consider related inequalities known as CHSH (Clauser, Horne, Shimony, and Holt) inequality which goes as follows. I will be following this pretty much as described in [1] and John Preskill's lecture notes (In fact a mixture of the two). Consider a hypothetical experiment in which Alice and Bob who have shared a physical system let us say two particles A and B that were produced in some experiment which Charlie is conducting. Charlie decided to send particle A to Alice and particle B to Bob who are very far apart. Now Alice is free to conduct experiments to measure two properties of A which we denote by Q and R both of which can have two possible measured values which we denote as $Q = \pm1, R = \pm1$. Similarly Bob also is allowed to choose from two properties S and T which also can have two possible values each $S, T = \pm1$. Note that $Q + R$ is either 0 or ±2. In either case following statement is true

$$QS + RS + QT - RT = (Q + R)S + (Q - R)T = \pm 2 \tag{1.1}$$

Now supposing there is a probability assigned to a particular outcome for each property $p(q, r, s, t)$ then the expectation or average value of measuring this particular combination is

$$\begin{aligned} |\langle QS + RS + QT - RT\rangle_c| &= |\sum_{q,r,s,t\epsilon\pm1} p(q, r, s, t)(qs + rs + qt - rt)| \\ &\leq 2 \sum_{q,r,s,t\epsilon\pm1} p(q, r, s, t) = 2 \end{aligned} \tag{1.2}$$

In other words

$$|\langle QS\rangle_c + \langle RS\rangle_c + \langle RT\rangle_c - \langle QT\rangle_c| \leq 2 \tag{1.3}$$

Let us now consider the following quantum state $|\psi^-\rangle$ of two spin $1/2$ particles prepared by Charlie who then sends one particle to Alice and one to Bob

$$|\psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}} \tag{1.4}$$

In this case we consider the properties we measure to be spin along different directions

$$\begin{array}{ll} Q = \vec{\sigma}^A.\hat{q} & S = \vec{\sigma}^B.\hat{s} \\ R = \vec{\sigma}^A.\hat{r} & T = \vec{\sigma}^B.\hat{t}. \end{array} \tag{1.5}$$

In the above equation, the superscript denotes on which particle the operator acts and $\vec{\sigma}$ denotes the following

$$\vec{\sigma} = \sigma_x\hat{x} + \sigma_y\hat{y} + \sigma_z\hat{z} \tag{1.6}$$

Now for Bell state $|\psi^-\rangle$ it is easy to check that

$$\left\langle \psi^- \middle| \left(\overrightarrow{\sigma}^{(A)} \cdot \hat{a}\right) \left(\overrightarrow{\sigma}^{(B)} \cdot \hat{b}\right) \middle| \psi^- \right\rangle = -\hat{a} \cdot \hat{b} = -\cos\theta \tag{1.7}$$

You can choose $\hat{q}, \hat{s}, \hat{r}, \hat{t}$ to be any co-planar directions separated by 45^0. For example choose

$$\hat{q} = \hat{x} \quad \hat{s} = \frac{\hat{x} + \hat{z}}{\sqrt{2}} \tag{1.8}$$

$$\hat{r} = \hat{z} \quad \hat{t} = \frac{-\hat{x} + \hat{z}}{\sqrt{2}}. \tag{1.9}$$

Substituting these in eq.(1.7) we get

$$\langle QS\rangle = \langle RS\rangle = \langle RT\rangle = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}, \tag{1.10}$$

$$\langle QT\rangle = -\cos\frac{3\pi}{4} = \frac{1}{\sqrt{2}} \tag{1.11}$$

Hence we have

$$|\langle QS\rangle_{Bell} + \langle RS\rangle_{Bell} + \langle RT\rangle_{Bell} - \langle QT\rangle_{Bell}| = 2\sqrt{2} \tag{1.12}$$

which clearly proves that the Bell state violates Bell's inequality which is based on the assumption of local realism. So either locality or realism is in trouble. In quantum field theory locality is crucial so henceforth for the purpose of these lectures we can give up on realism!

1.2 Definition

Consider a quantum system which is divided into two subsystems A and B. A given quantum state $|\psi\rangle_{AB}$ of this composite system is said to be an entangled state if it can not be expressed as the tensor product of the states of the subsystems

$$|\psi\rangle_{AB}^{ent} \neq |\psi_1\rangle_A \otimes |\psi_2\rangle_B \tag{1.13}$$

where as if one can do so then the state is said to be separable state. Let us demonstrate this with an example. Consider AB to be a composite system of two spin-$\frac{1}{2}$ systems. The state of A and B are given by

$$|\psi_1\rangle_A = \alpha_1 |1\rangle + \beta_1 |0\rangle \tag{1.14}$$
$$|\psi_2\rangle_B = \alpha_2 |1\rangle + \beta_2 |0\rangle \tag{1.15}$$

such that $|\alpha_i|^2 + |\beta_i|^2 = 1$. Therefore any separable state that can be constructed has to have the following form

$$|\psi\rangle_{AB}^{sep} = \alpha_1\alpha_2 |11\rangle + \beta_1\alpha_2 |01\rangle + \alpha_1\beta_2 |10\rangle + \beta_1\beta_2 |00\rangle \tag{1.16}$$

Now consider the quantum state (This is known as the Bell's state. We will know why in some time!)

$$|\tilde{\psi}\rangle_{AB} = a |11\rangle + b |00\rangle \tag{1.17}$$

Comparing eq.(1.16) and eq.(1.17) we see that there exist no α_1,α_2,β_1 and β_2 that can give us this state for non-zero a and b

$$\alpha_1\alpha_2 = a, \;\; \beta_1\beta_2 = b, \alpha_1\beta_2 = 0 \;\; and \;\; \beta_1\alpha_2 = 0 \tag{1.18}$$

Therefore the state mentioned in eq.(1.17) is an example of an entangled state. Instead of constructing Bell type inequalities one would like to have a general measure which will simply tell us the amount of entanglement present. This is where we will see that a quantity known as the entanglement measures becomes important.

1.3 Pure and mixed states

The maximum knowledge one can have in a quantum world is the quantum state or the wave function. If the exact quantum state of a given system is known then the system is said to be in a pure state. The density matrix for such a pure state is given by

$$\hat{\rho} = |\psi\rangle \langle\psi| \tag{1.19}$$

Therefore for a pure state Tr(ρ)=1 and the density matrix is the idempotent $\hat{\rho}^2 = \hat{\rho}$. The von-Neumann entropy for such a state is zero.

$$S = -Tr(\rho \log \rho) = -\sum_i \rho_i \log \rho_i = 0 \tag{1.20}$$

where ρ_is are the eigen values of the density matrix.

The mixed state as the name suggests is a mixture of several pure states with some weights and therefore has some classical probabilities along with the quantum uncertainity. It has the following form

$$\hat{\rho} = \sum_i p_i |\psi_i\rangle \langle\psi_i| \tag{1.21}$$

where p_i's are the classical probablities and $|\psi_i\rangle$s are the pure states. For a mixed state $Tr(\rho^2) < 1$. Example: Suppose we know that a state of spin-$\frac{1}{2}$ particle is prepared in a lab such that there is 50% probability that it is in the state $|\psi_1\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ and 50% probability that it is in the state $|\psi_2\rangle = |1\rangle$. Then the density matrix is given by

$$\hat{\rho} = \frac{1}{2}|\psi_1\rangle \langle\psi_1| + \frac{1}{2}|\psi_2\rangle \langle\psi_2| \tag{1.22}$$

1.4 Schmidt decomposition and reduced density matrix

Any quantum state of a bi-partite system whose Hilbert state is factorized $\mathcal{H}_A \otimes \mathcal{H}_B$ may be written as

$$|\psi\rangle = \sum_{jk} a_{jk} |j\rangle_A |k\rangle_B \tag{1.23}$$

According to the singular value decomposition theorem, any complex matrix can be decomposed into a product of diagonal matrix sandwitched between two unitary matrices

$$a_{jk} = \sum_{ijk} u_{ji} d_{ii} v_{ik} \tag{1.24}$$

Using this theorem one may always go into a basis where

$$|\psi\rangle_{AB} = \sum_i \lambda_i |i\rangle_A |i\rangle_B \tag{1.25}$$

where $|i\rangle_A = \sum_j u_{ji} |j\rangle_A$, $|i\rangle_B = \sum_k v_{ik} |k\rangle_B$ and $d_{ii} = \lambda_i$. This simplification for the bipartite quantum systems is known as the Schmidt decomposition.

1.5 Reduced density matrix

Suppose there is an observer who has access to only one of the subsystems say A, then his/her state is described by the reduced density matrix denoted as ρ_A which is obtained by tracing over the degrees of freedom of B

$$\rho_A = Tr_B(\hat{\rho}_{AB}) \tag{1.26}$$

where $\hat{\rho}_{AB}$ is the density matrix for the bi-partite system AB.

Using eq.(1.25), it is easy to see that for a pure state

$$\rho_A = \sum_k \langle k| \hat{\rho} |k\rangle_B = \sum_i \lambda_i^2 |i\rangle_A \langle i| \tag{1.27}$$

Therefore the reduced density matrix is diagonal in the Schmidt basis and the eigen values of the ρ_A are given by $|\lambda_i|^2$. Similarly it can be shown that ρ_B also has the same eigen values. The expectation value of an operator $O = O_A \otimes I$ is given by

$$\langle\psi| O |\psi\rangle = \sum_i |\lambda_i|^2 \langle i| O_A |i\rangle_A \tag{1.28}$$

$$= Tr(\rho_A O_A) \tag{1.29}$$

1.6 Entanglement entropy

For a bi-partite system in a pure state the amount of quantum entanglement can be quantified through a measure known as entanglement entropy denoted by S_A defined as the von-Neumann entropy of the reduced density matrix

$$S_A = -Tr(\rho_A \log(\rho_A)) \tag{1.30}$$

Example: Consider the state

$$|\psi\rangle = cos(\theta) |10\rangle + sin(\theta) |01\rangle \tag{1.31}$$

The reduced density matrix for the first particle which we are calling A is given by

$$S_A = -2cos^2(\theta) \log(cos(\theta)) - 2sin^2(\theta) \log(sin(\theta)) \tag{1.32}$$

Therefore we see that entanglement entropy is non-zero as long as $cos(\theta) \neq 0$ or $sin(\theta) \neq 0$ and it takes the maximum value at $\theta = \frac{\pi}{4}$ ($S_A^{max} = \log 2$) which is precisely the maximally entangled Bell state indicating that entanglement entropy is indeed a measure of entanglement for pure states.

1.7 Properties of entanglement entropy

1.7.1 Sub-additivity and mutual information

For a composite system $A \cup B$ the following inequality

$$S_A + S_B \geq S_{AB} \tag{1.33}$$

is known as the subadditivity property of the entanglement entropy. When the full system $A \cup B$ is in a pure state it is easy to see that the inequality is satisfied as $S_{AB} = 0$ and $S_A = S_B$. The difference between the L.H.S and the R.H.S of this inequality is a very important quantity known as the mutual information.

$$I(A:B) = S_A + S_B - S_{AB} \tag{1.34}$$

Mutual information is a measure of the total amount of correlations between A and B and therefore is a bound on the quantum entanglement between A and B. The subadditivity property can therefore be recast as $I(A:B) \geq 0$. When the full system is in a pure state $S_A = S_B$ (This is because ρ_A and ρ_B both have the same eigen values when $A \cup B$ is in a pure state) and $S_{AB} = 0$ therefore the inequality is trivially satisfied. For a given system the mutual information for an entangled state is always much larger than for any separable state.

1.7.2 Strong subadditivity

Another inequality that is obeyed by the entanglement entropies of different subsystems of a composite tripartite quantum system is known as the strong subadditivity property given by

$$S_{AB} + S_{BC} \geq S_{ABC} + S_B \tag{1.35}$$

This inequality can also be recast in terms of the mutual informations as

$$I(A:BC) \geq I(A:B) \tag{1.36}$$

which essentially says that the amount of correlation of the subsystem A with the subsystem $B \cup C$ will always be greater than that between A and B.

1.8 Renyi entropy

Renyi entropy of order 'n' is defined as follows

$$S_A^n = \frac{\ln[Tr((\rho^A)^n]}{1-n} \tag{1.37}$$

Where $n \in Z$. This quantity reduces to entanglement entropy in limit $n \to 1$.

$$\begin{aligned} S_A &= \lim_{n\to 1} S_A^n && (1.38) \\ &= -\lim_{n\to 1} \frac{\partial}{\partial n} \ln[Tr((\rho_A)^n] && (1.39) \\ &= -Tr(\rho_A \log \rho_A) && (1.40) \end{aligned}$$

Note that in order to take the limit in the last line it is required to analytically continue n through non-integer values. This analytic continuation is highly non-trivial has been proven to exist only for some handful of examples. We will see that this quantity is much more useful in computation of entanglement entropy in quantum field theory. Most of the topics covered in this section are from [1].

1.9 Basic Properties of Entanglement Measures

Following are some properties any entanglement measure has to satisfy

- Separable states contain no entanglement.. A state ρ_{AB} is said to be separable if and only if

$$\rho_{AB} = \sum_i p_i \rho_A^i \otimes \rho_B^i \tag{1.41}$$

  By definition separable state contains no entanglement and hence

$$E_{\rho_{AB}}(A:B) = 0 \tag{1.42}$$

- All non-separable states allow some tasks to be achieved better than by LOCC alone, hence all non- separable states are entangled.

- The entanglement of states does not increase under LOCC transformations. Any local operations done on A or B whether unitary or not can not increase the entanglement. A measure obeying this property is referred to as an entanglement montone under LOCC. Supposing you started with a state ρ of the full system which after LOCC goes to the state ρ_i with a classical probability p_i then

$$E_\rho(A:B) \geq \sum_i p_i E_{\rho_i'}(A:B) \tag{1.43}$$

- Entanglement does not change under Local Unitary operations.

$$E_\rho(A:B) = E_{\rho_i'}(A:B) \qquad \rho' = U_A \otimes U_B \, \rho \, (U_A \otimes U_B)^\dagger \tag{1.44}$$

- An entanglement measure is expected to obey monogamy. An entanglement measure is expected to obey the following (stronger) version of monogamy which is as follows

$$E(A:BC) \geq E(A:B) + E(A:C) \tag{1.45}$$

  A correlation measure such as the mutual information on the other hand is expected to obey a weaker version of monogamy which is as follows

$$E(A:BC) \geq E(A:B) \tag{1.46}$$

2 Mixed state and multipartite entanglement measures

Having discussed the information theoretic properties and significance of entanglement entropy in the previous lecture we now describe some other measures which characterize quantum correlations that are not directly available to entanglement entropy. We will restrict ourselves to two such measures which are computable in QFTs/CFTs.

2.1 Entanglement Negativity

Entanglement entropy is a valid entanglement measure for bi-partite systems in pure states only i.e $\rho_{AB} = |\psi\rangle\langle\psi|$. However for a generic mixed states where ρ_{AB} is mixed state one has to resort to other measures. One such computable measure is what is known as *entanglement negativity.* The definition of this measure proposed by Vidal and Werner [2] which was shown to be convex entanglement montone by [3] depends crucially on a criterion proposed by Asher Peres which we describe below.

2.1.1 Partial Transpose and Peres' PPT criterion

In order to understand Peres PPT criterion one first needs to define the operation of partial transpose which is as follows

$$\langle q_i^A q_j^B \mid \rho_{A\cup B}^{T_B} \mid q_k^A q_l^B \rangle = \langle q_i^A q_l^B \mid \rho_{A\cup B} \mid q_k^A q_j^B \rangle$$

where q^A, B are the basis states of the subsystems A and B respectively

- A separable mixed state was defined in the previous lecture has the following form

$$\rho_{AB} = \sum_i p_i \rho_A^i \otimes \rho_B^i \tag{2.1}$$

- Peres-Horodecki PPT criterion: is a necessary condition for the separability of a state. It states that for a separable state the eigen values remain positive under the operation of partial transpose (T_2) over one of the subsystems. This can be easily seen from the above equation the partial transpose simply acts as a transpose on ρ_B^i therefore leaving the eigen values invariant

- Quite interestingly the converse turns out be not true. A separable state is PPT but not all PPT states are separable. In other words it is a sufficient condition only for 2×2 and 2×3. Because negativity is based on this criterion it leaves out a class of entangled states referred to as "*bound entangled states*".

- *Entanglement Negativity* is defined as the trace norm of the partially transposed reduced density matrix of the bipartite system $A \cup B$

$$\mathcal{E} = \log ||\rho_{A\cup B}^{T_B}||.$$

- It is related to the absolute sum of the negative eigen values and therefore a measure of the extent to which *PPT criterion* is violated.

$$||\rho_{A\cup B}^{T_B}|| = \sum_i |\lambda_i| = 1 + 2 \sum_{\lambda_i<0} |\lambda_i|$$

- In quantum information literature $\mathcal{E}$ is referred to as the log negativity whereas the absolute sum of the negative eigen values is referred to as negativity

$$\mathcal{N} = \sum_{\lambda_i<0} |\lambda_i| = \frac{1}{2}(||\rho_{A\cup B}^{T_B}|| - 1)$$
$$\mathcal{E} = \log[1 + 2\mathcal{N}] \tag{2.2}$$

2.1.2 Properties and Examples

- A computable measure that provides the upper bound on the *distillable entanglement* which referes to the amount of Bell pairs you can extract from the state using only LOCC.
- It is an entanglement monotone i.e it does not increase under LOCC.
- Example: Let us take the $|GHZ\rangle$ and $|W\rangle$ state defined as

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$
$$\mathcal{E}(A:B) = \mathcal{E}(B:C) = \mathcal{E}(C:A) = 0$$

where as

$$|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$$
$$\mathcal{E}(A:B) = \mathcal{E}(B:C) = \mathcal{E}(C:A) = 0.34$$

2.2 Reflected Entropy and Entanglement of Purification

2.2.1 Purification

In order to understand the measures such as the reflected entropy and the entanglement of purification we must first understand the process of purification of a mixed state. Given a mixed state, process of purification involves constructing a pure state in a bigger Hilbert space such that after tracing over one of the subsystem leads to the reduced density matrix matches exactly with the mixed state in question.

$$\rho_{AB} = Tr_C\bigg(|\psi\rangle_{ABC}\langle\psi|\bigg) = \rho_{AB}^{mixed} \tag{2.3}$$

where ρ_{AB}^{mixed} is the mixed state in the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ and $|\psi\rangle_{ABC}$ is the pure state in the bigger Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$

The best example to get an intuition of the purification process is to consider the thermal state. Thermal state $\rho_\beta = e^{-\beta H}/Z$ is a mixed quantum state it can be easily checked that

$$\begin{aligned} Tr(\rho_\beta^2) &= \frac{1}{Z^2}\sum_n \langle n|e^{-2\beta H}|n\rangle = \frac{1}{Z^2}\sum_n e^{-2\beta E_n} \\ &= \sum_n p_n^2 < 1 \end{aligned} \tag{2.4}$$

where $Z = Tr(\rho_\beta)$ and $p_n = e^{-\beta E_n}/Z$ is simply the probability of the system to be in energy eigenstate E_n. The thermal state ρ_β can be purified by doubling the Hilbert space through what is known as the thermofield double state which is

$$|\psi\rangle_{TFD} = \frac{1}{Z}\sum_n e^{-\frac{\beta E_n}{2}}|n\rangle_L|n\rangle_R \tag{2.5}$$

It can be easily checked that tracing over either left or right part leads to the reduced density matrix given by the thermal state

$$\begin{aligned} \rho_L &= Tr_R(|\psi\rangle\langle\psi|) = \rho_\beta \\ \rho_R &= Tr_L(|\psi\rangle\langle\psi|) = \rho_\beta \end{aligned} \tag{2.6}$$

The above process of doubling the Hilbert space to purify can be performed on any mixed state not just on thermal state. For a generic mixed state the process is called canonical purification. Note that the process of purification is not unique in other words a given mixed state can have many possible purifications. Definition of reflected entropy is based on the canonical purification which is as follows

2.2.2 Canonical Purification

Definition

To begin with let us consider the mixed state ρ_{AB}. From quantum information theory we know that any mixed state could be expressed as a mixture orthonormal pure states $\rho^{(a)}_{AB} = |\phi_a\rangle\langle\phi_a|$ as follows

$$\begin{aligned} \rho_{AB} &= \sum_a p_a \rho^{(a)}_{AB} \\ &= \sum_a p_a |\phi_a\rangle\langle\phi_a| \end{aligned}$$

where p_a's are some classical probabilities such that $\sum_a p_a = 1$and $|\phi_a\rangle$'s correspond to orthonormal pure states. Each of these states $|\phi_a\rangle$ can be expressed in the Schmidt basis as follows

$$|\phi_a\rangle = \sum_i \sqrt{l^i_a}\, |i_a\rangle_A |i_a\rangle_B$$

where l_a are the eigen values, $|i_a\rangle_A$ and $|i_a\rangle_B$ are basis states of Hilbert spaces of A and B. This implies that

$$\rho_{AB} = \sum_{a,i,j} p_a \sqrt{l^i_a l^j_a}\, |i_a\rangle_A |i_a\rangle_B \langle j_a|_A \langle j_a|_B$$

The canonical purification for this mixed state is as follows

$$|\sqrt{\rho_{AB}}\rangle := \sum_{a,i,j} \sqrt{p_a l^i_a l^j_a}\, |i_a\rangle_A |i_a\rangle_B |j_a\rangle_{A^\star} |j_a\rangle_{B^\star}$$

It can be easily shown that tracing out A^* and B^* leads to the mixed state ρ_{AB} we started with

$$\begin{aligned} \rho_{ABA^*B^*} &= |\sqrt{\rho_{AB}}\rangle\langle\sqrt{\rho_{AB}}| \\ \mathrm{Tr}_{A^*B^*}(\rho_{ABA^*B^*}) &= \mathrm{Tr}_{A^*B^*}(|\sqrt{\rho_{AB}}\rangle\langle\sqrt{\rho_{AB}}|) \\ &= \rho_{AB} \end{aligned}$$

where we have used $\sum_j l_a^j = \mathrm{Tr}(|\phi_a\rangle\langle\phi_a|) = 1$. This proves that $|\sqrt{\rho_{AB}}\rangle$ is a purification of the mixed state ρ_{AB}.

The reflected entropy is defined as the Von-Neumann entropy of the subsystem AA^* of the canonically purified state $|\sqrt{\rho_{AB}}\rangle$

$$\boxed{S_R(A:B) := S(AA^*) = -\,\mathrm{Tr}_{AA^*}\left[\rho_{AA^*}\log\rho_{AA^*}\right]}$$

$$\rho_{AA^*} := \mathrm{Tr}_{BB^*}\,|\sqrt{\rho_{AB}}\rangle\langle\sqrt{\rho_{AB}}|$$

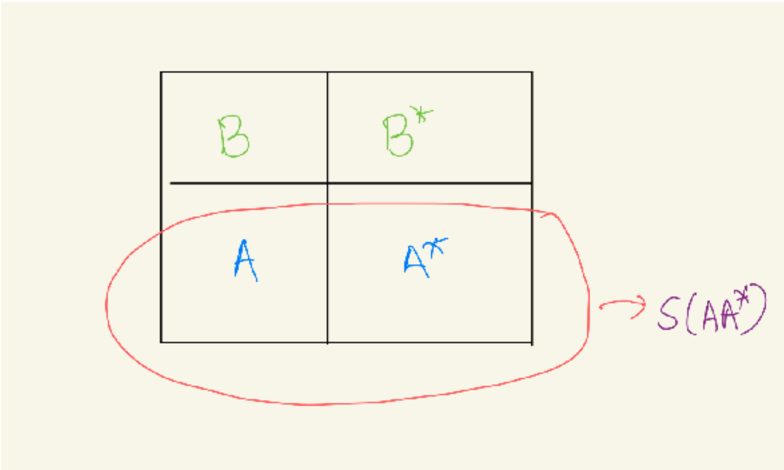


Figure 1: Reflected entropy of a bipartite system $A \cup B$

Quite interestingly this quantity was discovered not by quantum information theorists but by holographers!! The details can be found in [4] . We will be covering some of it in the present lectures.

Properties of Reflected Entropy

- Following equalities hold because A^*B^* is a copy of AB and $|\sqrt{\rho_{AB}}\rangle$ is pure

$$S(A) = S(A^*),\quad S(B) = S(B^*)\quad \text{and}\quad S(AA^*) = S(BB^*)$$

- When ρ_{AB} is a pure state

$$\rho_{AB} = |\psi\rangle\langle\psi| \implies S_R(A:B) = 2S(A) = 2S(B)$$

This is because it can be easily shown using the basis described above that for a pure state $\rho_{AA^*} = \rho_A \otimes \rho_A$.

- Vanishes for a tensor product state

$$\rho_{AB} = \rho_A \otimes \rho_B \implies S_R(A:B) = 0$$

This is because in this case $\rho_{AA^*BB^*} = \rho_{AA^*} \otimes \rho_{BB^*}$ and $|\sqrt{\rho_{AB}}\rangle = |\sqrt{\rho_A}\rangle \otimes |\sqrt{\rho_B}\rangle$

- However for a separable mixed state

$$\rho_{AB} = \sum_k p_k \rho_A^k \otimes \rho_B^k$$
$$S_R(A:B) = -\sum_k p_k \log p_k$$

Hence not a measure of mixed state entanglement and contains classical correlations. In contrast negativity which is measure of mixed state entanglement vanishes for such separable states.

- Bounded by Mutual Information. Start with strong subadditivity relation for AA^*B given by

$$S\left(AA^{\star}\right) - S\left(A^{\star}\right) + S(AB) - S(B) \geq 0$$

$$\implies S_R(A:B) \geq I(A:B)$$

- Bounded from below by

$$I\left(A:A^{\star}\right) = 2S(A) - S\left(AA^{\star}\right) \geq 0$$
$$I\left(B:B^{\star}\right) = 2S(B) - S\left(AA^{\star}\right) \geq 0$$

$$\implies S_R(A:B) \leq 2\min\{S(A), S(B)\}$$

- They attempted to prove monogamy. However could not prove

$$S_R(A:BC) \geq S_R(A:B)$$

For holographic states they were able to prove the above statement however for generic qubit states some are counter example states have been found recently [5].

2.3 Markov Gap

The importance of the Markov gap for reflected entropy as a measure of multipartite entanglement was analised in [6–8]

- Supposing there is a tripartite system ABC in some quantum state ρ_{ABC}. You know the reduced density matrix ρ_{AB} and you have access to only the B subsystem.

- *Markov recovery* process is about how accurately one can reconstruct the state ρ_{ABC} by operating say on B alone via a quantum map $\mathcal{R}_{B\to BC}$.

$$\tilde{\rho}_{ABC} = \mathcal{R}_{B\to BC}\left(\rho_{AB}\right)$$

- If $\mathcal{R}$ perfectly reproduces the original state ρ_{ABC} then it is called a quantum Markov map.

$$\tilde{\rho}_{ABC} = \rho_{ABC}$$

- However it was proven that

$$\max_{\mathcal{R}_{B\to BC}} F\left(\rho_{ABC}, \tilde{\rho}_{ABC}\right) \geq e^{-I(A:C|B)}$$

where $F(\rho,\sigma) = [\mathrm{Tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}]^2$ is the Fidelity which measures how close the original state is to the recovered state and $I(A:C \mid B) = S(AB)+S(BC)-S(ABC)-S(C)$ is the condition mutual information and it measures the strong subadditivity inequality.

- This seems like a bound on the maximum Fidelity one can obtain by performing an operation on subsystem B in terms of the conditional mutual information but one can reverse this relation to obtain

$$I(A:C \mid B) \geq -\max_{\mathcal{R}_{B\to BC}} \log F\left(\rho_{ABC}, \mathcal{R}_{B\to BC}\left(\rho_{AB}\right)\right)$$

In other words the above inequality gives the state dependent enhancement of strong subadditivity. As long as the Fidelity is not unity the above inequality implies that the strong subadditivity is bounded away from zero.

2.3.1 Markov Gap of Reflected Entropy

- Let us now consider the difference between the reflected entropy and mutual information

$$S_R(A:B) - I(A:B) = I\left(A:B^* \mid B\right) = I\left(B:A^* \mid A\right)$$

- The above described theorem implies a stronger equality than $S_R(A:B) \geq I(A:B)$ which was proven earlier

$$S_R(A:B) - I(A:B) \geq -\max_{\mathcal{R}_{B\to BB^*}} \log F\left(\rho_{ABB^*}, \mathcal{R}_{B\to BB^*}\left(\rho_{AB}\right)\right)$$
$$S_R(A:B) - I(A:B) \geq -\max_{\mathcal{R}_{A\to AA^*}} \log F\left(\rho_{AA^*B}, \mathcal{R}_{A\to AA^*}\left(\rho_{AB}\right)\right)$$

2.3.2 Markov gap and multipartite entanglement

- Multipartite entanglement is subtle and interesting.

- Simplest examples are the tripartite entangled states: W-state and GHZ. Entanglement structure of W-state is very different from the GHZ state

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$
$$|\mathrm{W}\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$$

- The entanglement structure of GHZ is entirely tripartite i.e if you trace over one of the qubit there is no entanglement between the remaining subsystems. However, the W-state has some entanglement even after we trace over one of the qubit.

- consider the state $|W_\epsilon\rangle$ defined as

$$|W_\epsilon\rangle = \epsilon|100\rangle + \epsilon|001\rangle + \sqrt{1-2\epsilon^2}|010\rangle$$

which goes from a pure product state $\epsilon = 0$ to W state for $\epsilon = \frac{1}{\sqrt{3}} = 0.577$ to another unentangled state for $\epsilon = \frac{1}{\sqrt{2}} = 0.707$

- As can be seen from the graph below the Markov gap for canonical purification computed from reflected entropy goes to maximum as one reaches the $|\mathrm{W}\rangle$ state where as for GHZ state the Markov gap simply vanishes

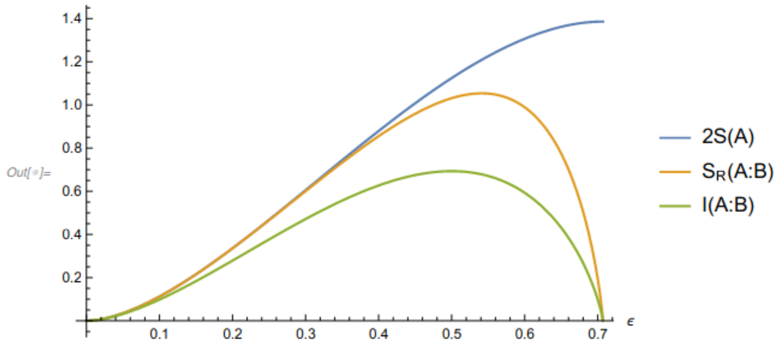


Figure 2: Markov gap for canonical purification of the subsystem AB obtained from $|W_\epsilon\rangle$. Picture from [9]

- Hence, the expectation was that the existence of Markov gap points towards the presence of the multipartite W-type entanglement in the system. This has now been proven to be true in [8].

3 Entanglement Entropy in Quantum Field Theories

3.1 Density matrix in the pathintegral formalism

The groundstate wavefunction of a non-relativistic quantum particle is defined in the Euclidean path integral formalism by integrating over all possible paths from $t_E = -\infty$

$$\psi_0(x) = \mathcal{N} \int_{x(t_E=-\infty)}^{x(t_E=0)} \mathcal{D}x \;\; e^{-S[x]} \tag{3.1}$$

The reason for this is as follows. Consider the transition amplitude of a particle from an initial point (x_i, t_i) to a final point(x_f, t_f)

$$\langle x_f, t_f | x_i, t_i \rangle = \int_{x(t_i)}^{x(t_f)} \mathcal{D}x \;\; e^{iS[x]},$$
$$\langle x_f | e^{iH(t_f - t_i)} | x_i \rangle = \int_{x(t_i)}^{x(t_f)} \mathcal{D}x \;\; e^{iS[x]}$$

Performing a wick rotation $t \to it_E$ and choosing the initial and final positions to be $x(t_E = -\tau)$ and $x(t_E = 0)$ respectively and then inserting a complete set of states we get

$$\sum_n \langle x_f | e^{-H(\tau)} | \, |n\rangle \, \langle n| \, |x_i\rangle = \int_{x(t_E=-\tau)}^{x(t_E=0)} \mathcal{D}x \;\; e^{-S_E[x]} \tag{3.2}$$

As $\tau \to \infty$ or $t_E \to -\infty$ one can see from the L.H.S of the above equation that the only contribution comes from the ground state therefore we have

$$e^{-E_0(\tau)} \langle x_f \, |0\rangle \, \langle 0| \, x_i \rangle = \int_{x(t_E=-\infty)}^{x(t_E=0)} \mathcal{D}x e^{-S_E[x]}$$
$$\psi_0(x) = \frac{e^{E_0(\tau)}}{\langle 0| \, x_i \rangle} \int_{x(t_E=-\infty)}^{x(t_E=0)} \mathcal{D}x e^{-S_E[x]}$$

This implies that upto a normalization factor the ground state wave function is indeed given by eq.(3.1). The analogue of this in relativistic quantum field theory is the vacuum wave functional which is obtained by integrating over all field configurations in the lower half of the Euclidean plane given by

$$\Psi_0[\phi] = \mathcal{N} \int_{t_E=-\infty}^{\phi(x), t_E=0} [\mathcal{D}\phi] \;\; e^{-S_E[\phi(x)]} \tag{3.3}$$

Similarly the conjugate wave functional is defined by integrating over all the field configurations on the upper half plane

$$\Psi_0^*[\phi] = \mathcal{N} \int_{\phi(x), t_E=0}^{t_E=\infty} [\mathcal{D}\phi] \;\; e^{-S_E[\phi(x)]} \tag{3.4}$$

Therefore the matrix element of the density operator is obtained by integrating over the full Euclidean space with the boundary conditions provided at $\phi(x, 0^-) = \phi_-(x)$ and $\phi(x, 0^+) = \phi_+(x)$

$$\rho_{\phi_- \phi_+} = \frac{1}{Z} \int_{t_E=-\infty}^{t_E=\infty} [\mathcal{D}\phi] \;\; e^{-S_E[\phi(x)]} \prod_x \delta(\phi(0^+, x) = \phi_+(x)) \;\; \delta(\phi(0^-, x) = \phi_-(x)) \tag{3.5}$$

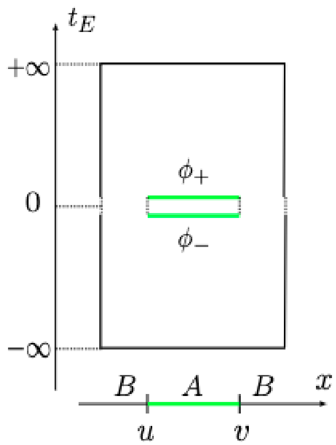


Figure 3: Picture from [10]

The factor Z is introduced to ensure that $Tr(\rho) = 1$ and it is equal to the Vacuum partition function. Suppose we divide the full system into subsystem-A(which we choose to be an interval $[u, v]$ and the rest of the system is denoted as B. The reduced density matrix of a subsystem A is then obtained by taking a partial trace which for this case implies that one has to identify $\phi_+(x) = \phi_-(x)$ for $x \in B$ and integrating over ϕ_+

$$[\rho_A]_{\phi_-\phi_+} = \frac{1}{Z_1}\int_{t_E=-\infty}^{t_E=\infty} [\mathcal{D}\phi] \;\; e^{-S_E[\phi(x)]} \prod_{x\in A} \delta(\phi(0^+, x) = \phi_+(x)) \;\; \delta(\phi(0^-, x) = \phi_-(x))$$

One can see from above equation that obtaining the entanglement entropy using $S_A = -Tr(\rho_A \log \rho_A)$ is extremely difficult for a quantum field theory as it involves the evaluation of logarithm of the above mentioned reduced density matrix. However for a (1+1) dimensional quantum field theory with conformal symmetry one can to resort to a technique known as the replica technique which was developed by Cardy et. al in [11, 12].

3.2 The replica technique

The replica technique as the name suggests involves the replication of n-copies of the theory and evaluation of the entanglement entropy through the Renyi entropy in the $n \to 1$ limit (known as the replica limit). The trace of ρ_A^n is found by preparing n copies, making the identification $\phi_+^i(x) = \phi_-^{(i+1)}(x)$ and integrating over these variables

$$Tr_A(\rho_A^n) = \frac{1}{Z_1^n}\int_{(x,t_E)\in\mathcal{R}} [\mathcal{D}\phi] \;\; e^{-S_E[\phi(x)]} = \frac{Z_n(A)}{Z_1^n} \tag{3.6}$$

where $Z_n(A)$ is now the partition function on the n-sheeted Riemann surface. This complicated path integral on the Riemann surface can be mapped to a multi-copy

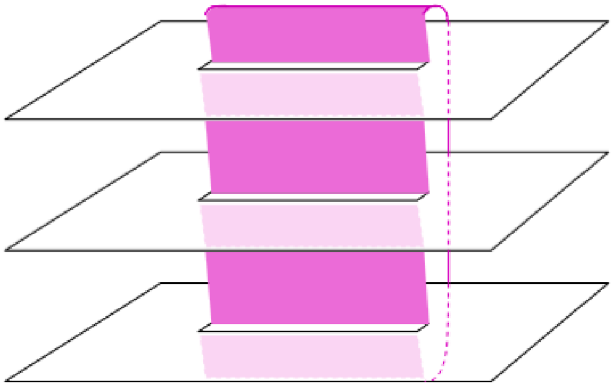


Figure 4: Picture from [13]

model on the complex plane provided the boundary conditions $\phi^i_+(x) = \phi^{(i+1)}_-(x)$ are satisfied i.e for every field ϕ there will be n-fields $(\phi_1, \phi_2 ... \phi_n)$ which obey these conditions (here the superscripts i and $(i+1)$ indicate the fields on i^{th} and $(i+1)^{th}$ Riemann sheet respectively).

$$Z_n(A) = \int_{\mathcal{C}} [d\phi_1 d\phi_2 .... d\phi_n] \exp[- \int dx dt_E \; [\mathcal{L}^{(1)} + \mathcal{L}^{(2)} + ... + \mathcal{L}^{(n)}] \tag{3.7}$$

This path integral is restricted by the above mentioned boundary conditions and these conditions are imposed by the local fields called the branch point twist fields placed at the end point of the subsystem$A$. The operations of these *twist* and *anti twist* fields are defined as follows

$$\begin{aligned} \mathcal{T}_n(u) &: \phi^{(i)}(x) \to \phi^{(i+1)}(x) \quad for \; x > u \\ \bar{\mathcal{T}}_n(v) &: \phi^{(i)}(x) \to \phi^{(i-1)}(x) \quad for \; x > v \end{aligned}$$

where the index i has to be understood as i mod n. Therefore, any correlation function between the fields on the Riemann surface is equal to the correlation function of the replicated fields along with the branch point twist fields on the complex plane

$$\left\langle O(x, t_E : i^{th} sheet) ... \right\rangle_{\mathcal{L}_{\mathcal{R}}} = \frac{\left\langle \mathcal{T}_n(u) \bar{\mathcal{T}}_n(v) O_i(x, t_E) ... \right\rangle_{\mathcal{L}_c^{(n)}}}{\left\langle \mathcal{T}_n(u) \bar{\mathcal{T}}_n(v) \right\rangle_{\mathcal{L}_c^{(n)}}} \tag{3.8}$$

This leads us to a relation between the partition function on the n- sheeted Riemann surface and the two point correlation function of the twist and the anti-twist fields given by

$$Z_n(A) \propto \left\langle \mathcal{T}_n(u) \bar{\mathcal{T}}_n(v) \right\rangle_{\mathcal{L}_c^{(n)}} \tag{3.9}$$

This may be generalized to a scenario where the subsystem-A involves union N-disjoint intervals $\cup[u_i, v_i]$ $(i = 1, 2...N)$ as follows

$$Z_n(A) \propto \langle \mathcal{T}_n(u_1)\bar{\mathcal{T}}_n(v_1)\mathcal{T}_n(u_2)\bar{\mathcal{T}}_n(v_2).....\mathcal{T}_n(u_N)\bar{\mathcal{T}}_n(v_N) >_{\mathcal{L}_c^{(n)}} \tag{3.10}$$

3.3 Twist operators

Consider the conformal map $z = (\frac{w-u}{w-v})^{\frac{1}{n}}$ that maps the w-coordinates on the n-sheeted Riemann surface to the z-coordinates on the complex plane. Under this map stress energy tensor $T(w)$ transforms in the following way

$$T(w) = \frac{dz}{dw}T(z) + \frac{c}{12}\{z, w\} \tag{3.11}$$

We will use this transformation to argue that the twist operators are the primary operators of the conformal field theory and determine the scaling dimension. Taking the expectation value on both sides and using the fact that on the complex plane $\langle T(z) \rangle_{\mathbf{C}} = 0$ (because of the translational and rotational invariance) we get

$$\langle T(w) \rangle_{\mathcal{R}} = \frac{c}{24}\frac{(1 - \frac{1}{n^2})(v-u)^2}{(w-u)^2(w-v)^2} \tag{3.12}$$

One can now use the conformal Ward identity to argue that the above function has the expected form in (3.8)

$$\langle T(w) \rangle_{\mathcal{R}} = \frac{\langle \mathcal{T}_n(u)\bar{\mathcal{T}}_n(v)T_i(z) \rangle_{\mathbf{c}}}{\langle \mathcal{T}_n(u)\bar{\mathcal{T}}_n(v) \rangle_{\mathbf{c}}} \tag{3.13}$$

only if the twist and the anti-twist operators assumed to be the primary operators of the scaling dimension given by

$$\Delta_n = \frac{c}{12}(n - \frac{1}{n}) \tag{3.14}$$

This is because the conformal ward identity is given by

$$\langle \mathcal{T}_n(u)\bar{\mathcal{T}}_n(v)T(z) \rangle_{\mathbf{c}} = \left(\frac{1}{w-u}\frac{\partial}{\partial u} + \frac{1}{w-v}\frac{\partial}{\partial v} + \frac{h_n}{(w-u)^2} + \frac{h_n}{(w-v)^2} \right) \langle \mathcal{T}_n(u)\bar{\mathcal{T}}_n(v) \rangle_{\mathbf{c}}$$

Since the form of the two point correlation function of the primary operators in a CFT is fixed to be

$$\langle \mathcal{T}_n(u)\bar{\mathcal{T}}_n(v) \rangle_{\mathbf{c}} = \frac{c_n}{|u-v|^{2\Delta_n}} \tag{3.15}$$

one gets eq.(3.13) if we assume the scaling dimension to be that given in eq.(3.14).

3.4 EE of a single interval in CFT$_{1+1}$

3.4.1 Vacuum state

Finally we have all the ingredients required to evaluate the EE of a (1+1) dimensional conformal field theory . From eq.(3.9) and eq.(3.6) we already know that

$$Tr(\rho_A^n) = \left\langle \mathcal{T}_n(u)\bar{\mathcal{T}}_n(v)\right\rangle_{\mathbf{c}} = \frac{c_n}{|u-v|^{2\Delta_n}} \tag{3.16}$$

Therefore from the definition of Renyi entropy in section 1.6 and eq.(3.16) entanglement entropy may obtained in the replica limit from the two point correlation function of the twist and the anti-twist operators

$$\begin{aligned} S_A &= \lim_{n\to 1}\frac{\partial}{\partial n}\ln[\left\langle \mathcal{T}_n(u)\bar{\mathcal{T}}_n(v)\right\rangle_{\mathbf{c}}] & (3.17)\\ &= \frac{c}{3}\log[\frac{l}{a}] + c_1 & (3.18)\end{aligned}$$

where $\ell = |u-v|$ is the length of the subsystem-A, c_1 is a constant and a-is the UV cut-off of the field theory introduced to make the quantity inside the log dimension less as it is the only other dimension-full parameter in the theory.

3.4.2 Finite temperature case

The finite temperature EE of the single interval may be obtained by using the conformal map from the complex plane to cylinder ($w = \frac{\beta}{2\pi}\log[z]$) as the two point correlation function transforms in the following way

$$\left\langle \mathcal{T}_n(w_1)\bar{\mathcal{T}}_n(w_2)\right\rangle_{\beta} = \prod_{i=1}^{4}|\frac{dz_i}{dw_i}|^{\Delta_n^i}\left\langle \mathcal{T}_n(z_1)\bar{\mathcal{T}}_n(z_2)\right\rangle_{\mathbf{C}} \tag{3.19}$$

which leads to following expression for entanglement entropy

$$S_A = \frac{c}{3}\log\left[\frac{\beta}{\pi a}\sinh\left(\frac{\pi\ell}{\beta}\right)\right] + c_1' \tag{3.20}$$

Note that unlike the Vacuum case where the full system was in a pure state the finite temperature state is an example of a mixed state. As a result entanglement entropy contains contribution from both the thermal and quantum correlations. In fact at high temperature all the quantum correlations disappear. This may be observed by taking the $\beta \to 0$ limit in the above equation

$$S_A \to \frac{\pi c\ell}{3\beta} \tag{3.21}$$

and therefore receives a dominant contribution from the thermal entropy of the subsystem-A as the thermal entropy density of a CFT_{1+1} is $s = \frac{\pi c}{3\beta}$.

4 Entanglement Negativity in Quantum Field Theories

In this section we briefly review the computation of the entanglement negativity for mixed states described by two different configurations in a CFT_{1+1} relevant for our purpose. In this context we first introduce the definition of entanglement negativity in quantum information theory as proposed in [2] which was shown to be an entanglement monotone in [3]. The authors there considered a tripartite system in a pure state consisting of subsystems that are denoted as A_1, A_2 (such that $A_1 \cup A_2 = A$) and A^c describing the rest of the system. The entanglement negativity characterizing the upper bound on the *distillable entanglement* between the subsystems A_1 and A_2 is defined as follows

$$\mathcal{E} = \log \mathrm{Tr}|(\rho_A^{T_2})| \tag{4.1}$$

where the ρ_A is the reduced density matrix of the subsystem $A = A_1 \cup A_2$ and the superscript T_2 indicates the operation of partial transpose which is defined as follows

$$\langle e_i^{(1)} e_j^{(2)} | \rho^{T_2} | e_k^{(1)} e_l^{(2)} \rangle = \langle e_i^{(1)} e_l^{(2)} | \rho | e_k^{(1)} e_j^{(2)} \rangle. \tag{4.2}$$

Here $|e_i^{(1)}\rangle$ and $|e_j^{(2)}\rangle$ represent the basis states of the subsystems A_1 and A_2 respectively.

As discussed in the introduction, Calabrese et al. in [14**?** , 15] developed a replica technique to compute the entanglement negativity for various pure and mixed state configurations in a CFT_{1+1}. The first configuration depicted in fig.(5) involves the subsystems A_1 and A_2 corresponding to two disjoint finite intervals denoted as$[u_1, v_1]$ and $[u_2, v_2]$ respectively and A^c describes the rest of the system. The replica definition of the entanglement negativity between the subsystems A_1 and A_2 is given as follows

$$\mathcal{E} = \lim_{n_e \to 1} \log \mathrm{Tr}(\rho_{AB}^{T_B})^{n_e}. \tag{4.3}$$

Here n_e denotes that the parity of the replica index n is even. Note that this is because the definition of entanglement negativity given in eq.(4.1) matches with the above definition only if we assume the even parity of n that is $n = n_e$ and then finally take the limit $n_e \to 1$. Therefore the authors proposed the replica definition for the entanglement negativity as an analytic continuation of even sequences of n to $n_e = 1$.

4.1 Pure state

For a pure state entanglement negativity is given by Renyi entropy of order half. The proof is as follows

$$|\Psi\rangle = \sum_j c_j \left| e_j^{(A)} e_j^{(B)} \right\rangle \tag{4.4}$$

The density matrix is therefore given by

$$\rho_{AB} = \sum_{j,k} c_j c_k |e_j^{(A)} e_j^{(B)}\rangle\langle e_k^{(A)} e_k^{(B)}|, \quad \rho_B = \mathrm{Tr}_A(\rho) = \sum_j c_k^2 e_k^{(B)}\rangle\langle e_k^{(B)}| \tag{4.5}$$

Hence

$$\rho_{AB}^{T_B} = \sum_{j,k} c_j c_k |e_j^{(A)} e_k^B\rangle\langle e_k^{(A)} e_j^{(B)}| \tag{4.6}$$

This leads to

$$(\rho_{AB}^{T_B})^n = \{ \begin{matrix} \sum_{j,k} c_j^{n_o} c_k^{n_o} e_k^{(B)} e_j^{(A)}\rangle\langle e_j^{(B)} e_k^{(A)}|, \; n = n_o \text{ odd} \\ \sum_{j,k} c_j^{n_e} c_k^{n_e} e_k^{(B)} e_j^{(A)}\rangle\langle e_k^{(B)} e_j^{(A)}|, \; n = n_e \text{ even} \end{matrix} . \tag{4.7}$$

Which in turn leads to

$$\mathrm{Tr}\left(\rho_{AB}^{T_B}\right)^n = \begin{cases} \sum_r c_r^{2n_o} = \mathrm{Tr}\, \rho_A^{n_o} \\ [\sum_r c_r^{n_e}]^2 = \left(\mathrm{Tr}\, \rho_A^{n_e/2}\right)^2 \end{cases} \tag{4.8}$$

Hence we have

$$\mathcal{E} = S_A^{(1/2)} = 2\log[Tr(\rho_A^{1/2})] \tag{4.9}$$

4.2 Mixed state

The authors demonstrated that the quantity $\mathrm{Tr}(\rho_A^{T_2})^{n_e}$ in eq.(4.3) is given by the following four point twist correlator

$$\mathrm{Tr}(\rho_{AB}^{T_B})^{n_e} = \langle \mathcal{T}_{n_e}(u_1)\overline{\mathcal{T}}_{n_e}(v_1)\overline{\mathcal{T}}_{n_e}(u_2)\mathcal{T}_{n_e}(v_2)\rangle_{\mathbb{C}}. \tag{4.10}$$

where $\mathcal{T}$ and $\overline{\mathcal{T}}$ are the twist and the anti-twist operators both of which have the scaling dimensions $\Delta_{n_e} = \frac{c}{24}(n_e - \frac{1}{n_e})$.

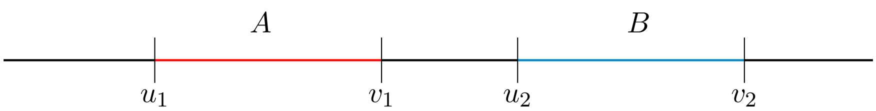


Figure 5: Schematic of the configuration of the mixed state of two disjoint intervals A_1 and A_2

Note that the four point correlator in a CFT_{1+1} can only be fixed upto a function of the cross ratio $\left[x = \frac{(u_1-v_1)(u_2-v_2)}{(u_1-u_2)(v_1-v_2)}\right]$, which depends on the full operator content of the theory. However, in the limit of adjacent intervals described as $v_1 \to u_2$ as depicted in fig.(6), the four point twist correlator in eq.(4.10) reduces to the following three point twist correlator as follows

$$\mathrm{Tr}(\rho_A^{T_2})^{n_e} = \langle \mathcal{T}_{n_e}(u_1)\overline{\mathcal{T}}^2_{n_e}(u_2)\mathcal{T}_{n_e}(v_2)\rangle_{\mathbb{C}}. \tag{4.11}$$

where $\overline{\mathcal{T}}^2_{n_e}(u_2)$ corresponds to the twist operator which connects j^{th}-sheet of the Riemann surface to $(j-2)^{th}$-sheet and has the following scaling dimension

$$\Delta_{n_e}^{(2)} = 2\Delta_{\frac{n_e}{2}} = \frac{c}{12}(\frac{n_e}{2} - \frac{2}{n_e}.) \tag{4.12}$$

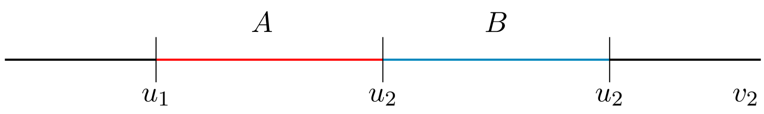


Figure 6: Schematic of the configuration in the limit the two intervals A_1 and A_2 become adjacent

The form of the three point function may be completely determined by the conformal symmetry up to a numerical constant (which also depends on the full operator content) and leads to the following expression for the entanglement negativity [14**?**] of the mixed state in the CFT_{1+1} at zero temperature.

$$\mathcal{E} = \frac{c}{4}\log[\frac{l_1 l_2}{(l_1+l_2)a}] + constant \tag{4.13}$$

where l_1 and l_2 are the lengths of the two intervals, c is the central charge of the CFT_{1+1} and a is the UV cut-off for the CFT_{1+1}.

Subsequently the authors addressed the case of a bipartite system in a pure state, described by a single interval in a zero temperature CFT_{1+1}. The entanglement negativity for this configuration is obtained by taking the bipartite limit defined by $u_2 \to v_1$ and $v_2 \to u_1$ (that is $(A, B, (AB)^c) \to (A, A^c, \emptyset)$) in the four point function given by eq.(4.10) which leads to the following expression

$$\mathrm{Tr}(\rho_A^{T_2})^{n_e} = \langle \overline{\mathcal{T}}^2_{n_e}(u)\mathcal{T}^2_{n_e}(v)\rangle_{\mathbb{C}} \tag{4.14}$$

where $u_1 = v_2$ is denoted as u and $v_2 = u_1$ is denoted as v. Since the form of the two point correlation function is fixed completely by the conformal symmetry, the entanglement negativity may be easily computed using eq.(4.3) as[1]

$$\mathcal{E} = \frac{c}{2}\log\left(\frac{l}{a}\right) + constant, \tag{4.15}$$

where c is the central charge, $\ell = |u-v|$ is the length of the subsystem-A and a represents the UV cut-off of the field theory. Interestingly this limit leads to the expected result from quantum information that for a pure state the entanglement negativity is Renyi entropy of order-$\frac{1}{2}$.

Although the above procedure works for the pure vacuum state of the CFT_{1+1}, eq.(4.14) is not applicable to the finite temperature mixed state where the CFT_{1+1} is defined on an infinite cylinder [15]. For the latter case the bipartite limit is more subtle and involves the full tripartite system. This configuration involves the subsystems A, B_1, B_2 described by the intervals $A = [u_2, v_2]$ of length ℓ, $B_1 = [u_1, v_1]$

[1]Note that the twist fields $\mathcal{T}^2_{n_e}$ connect n_e^{th} sheet of the Riemann surface $(n_e+2)^{th}$($\overline{\mathcal{T}}^2_{n_e}$ connect n_e^{th} to $(n_e-2)^{th}$) sheet of the Riemann surface. This leads to the factorization of the two point function due to the breaking of n_e even sheeted Riemann surface into two $n_e/2$ sheeted Riemann surfaces.

and $B_2 = [u_3, v_3]$ as depicted in the fig.(7) below. We denote $B = B_1 \cup B_2$ and A^c describes the rest of the system.

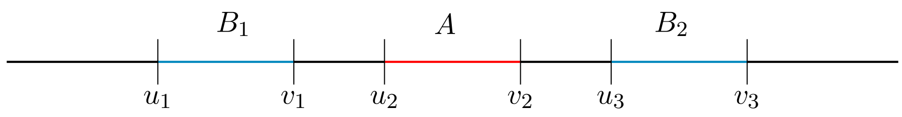


Figure 7: Schematic of the configuration with single interval A in between two disjoint intervals B_1 and B_2

In this case the entanglement negativity is characterized by the six point function of the twist fields as follows

$$\mathrm{Tr}(\rho_{AB}^{T_A})^{n_e} = \langle \mathcal{T}_{n_e}(u_1)\overline{\mathcal{T}}_{n_e}(v_1)\overline{\mathcal{T}}_{n_e}(u_2)\mathcal{T}_{n_e}(v_2)\mathcal{T}_{n_e}(u_3)\overline{\mathcal{T}}_{n_e}(v_3)\rangle \tag{4.16}$$

In the limit $v_1 \to u_2$ and $v_2 \to u_3$ the configuration in fig.(7) reduced to the one in fig.(8) and the above six point function reduces to the following four point function

$$\mathrm{Tr}(\rho_{AB}^{T_A})^{n_e} = \big\langle \mathcal{T}_{n_e}(u_1)\overline{\mathcal{T}}^2_{n_e}(u_2)\mathcal{T}^2_{n_e}(v_2)\overline{\mathcal{T}}_{n_e}(v_3)\big\rangle \tag{4.17}$$

For the vacuum state of the CFT_{1+1} which lives on the complex plane, the above four point twist correlator has the following form from conformal symmetry [15]

$$\big\langle \mathcal{T}_{n_e}(z_1)\overline{\mathcal{T}}^2_{n_e}(z_2)\mathcal{T}^2_{n_e}(z_3)\overline{\mathcal{T}}_{n_e}(z_4)\big\rangle_{\mathbb{C}} = \frac{c_{n_e}c^2_{n_e/2}}{z_{14}^{2\Delta_{n_e}}\, z_{23}^{2\Delta^{(2)}_{n_e}}}\frac{\mathcal{F}_{n_e}(x)}{x^{\Delta^{(2)}_{n_e}}}, \qquad x \equiv \frac{z_{12}z_{34}}{z_{13}z_{24}}, \tag{4.18}$$

where $(z_1, z_2, z_3, z_4) = (u_1, u_2, u_3, v_3)$ for the configuration in question. This leads to the following expression for the entanglement negativity of the mixed state configuration depicted in fig.(8)

$$\mathcal{E} = \frac{c}{4}\log\Big(\frac{l_1\, l_2^2\, l_3}{(l_1+l_2)(l_2+l_3)a^2}\Big) + g(x) + \text{constant}, \quad x = \frac{l_1 l_3}{(l_1+l_2)(l_2+l_3)} \tag{4.19}$$

where $l_1 = |u_1 - u_2|$, $l_2 = |u_2 - v_2|$ and $l_3 = |v_2 - v_3|$ are the lengths of the intervals B_1, A and B_2 respectively. The function $g(x)$ and the constant are non universal and depend on the full operator content of the theory. However the end point values of the function $g(x)$ may be fixed to be $g(1) = 0$ and $g(0) = const$ as described in [15].

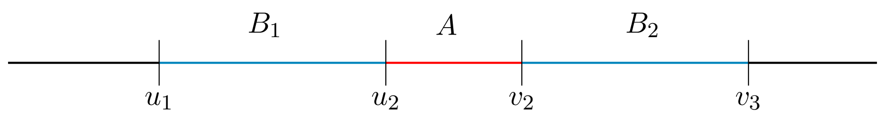


Figure 8: Schematic of the configuration in the limit the two intervals B_1 and B_2 become adjacent to A

Interestingly, observe that in the limit $l_1, l_3 >> l_2$, i.e as the cross ratio $x \to 1$, the tripartite system involving A, B and the rest of the system reduces to the bipartite system $A \cup A^c$. Note that this limit is equivalent to the bipartite limit $B \to A^c$ leading to $(A \cup B)^c \to \emptyset$. Hence in this limit, the result given by eq.(4.19) reduces to the following

$$\mathcal{E} \approx \frac{c}{2} \log\left(\frac{l_2}{a}\right) + \text{constant}. \tag{4.20}$$

This expression matches exactly with the entanglement negativity of the bipartite pure vacuum state of the CFT_{1+1} given in eq.(4.15). This is expected as in the bipartite limit $B \to A^c$, the mixed state density matrix $\rho_{A\cup B}$ reduces to the pure vacuum state $\rho_{A\cup A^c} = |0\rangle\langle 0|$ for the case when the full system $A \cup A^c$ is described by the vacuum state of the CFT_{1+1}.

Similarly, as described in [15], the entanglement negativity of the finite temperature mixed state in the bipartite limit is described through the four point twist correlator as follows

$$\mathcal{E} = \lim_{L\to\infty} \lim_{n_e\to 1} \log\left[\left\langle \mathcal{T}_{n_e}(-L)\overline{\mathcal{T}}^2_{n_e}(-l)\mathcal{T}^2_{n_e}(0)\overline{\mathcal{T}}_{n_e}(L)\right\rangle_\beta\right], \tag{4.21}$$

Once again, the full tripartite system depicted in fig(8), reduces to the bipartite system $A \cup A^c$ in the limit $B \to A^c$ ($L \to \infty$), with the coordinates $u_2 = -L$ and $v_3 = L$ in eq.(4.17). where the subscript β indicates that the correlation function has to be evaluated on a cylinder, $v_2 = u_1 = \ell$ and $v_1 = u_3 = 0$. Note that the order of the limits is significant to obtain the correct finite temperature negativity, and unlike the zero temperature case discussed previously, the full system $A \cup A^c$ in this case is described by the mixed state thermal density matrix $\rho_{A\cup A^c} = e^{-\beta H}$.

As described by eq.(4.21), the replica limit $n_e \to 1$ has to be imposed prior to the bipartite limit denoted by $L \to \infty$. The four point function of the primary operators is fixed only up to a function of a cross ratios and therefore the entanglement negativity given by eq.(4.21) leads to the following expression

$$\mathcal{E} = \frac{c}{2} \log\left[\frac{\beta}{\pi a} \sinh\left(\frac{\pi l}{\beta}\right)\right] - \frac{\pi c l}{2\beta} + g(e^{-2\pi l/\beta}) + constant. \tag{4.22}$$

The non universal function $g(e^{-2\pi l/\beta})$ and the constant in the above expression depend on the full operator content of the theory. Its values may be fixed only at the end points ($x = 0$ and $x = 1$). Interestingly the above expression may be expressed as follows

$$\mathcal{E} = \frac{3}{2}[S_A - S_A^{th}] + g(e^{-2\pi l/\beta}) + constant, \tag{4.23}$$

where $S_A = \frac{c}{3} \log\left[\frac{\beta}{\pi a} \sinh\left(\frac{\pi l}{\beta}\right)\right]$ and $S_A^{th} = \frac{\pi c l}{3\beta}$ corresponds to the entanglement entropy and the thermal entropy of the subsystem-A respectively. It is clear from the above expression that the entanglement negativity eliminates the thermal contribution and hence describes the upper bound on the distillable entanglement in a mixed state.

5 Reflected Entropy in Quantum Field Theories

Having discussed the replica technique to compute entanglement entropy in quantum field theories, we now proceed to describe similar techniques to compute other measures. We begin by describing the replica technique for reflected entropy as described in [4].

5.1 Replica Technique: Renyi Reflected Entropy

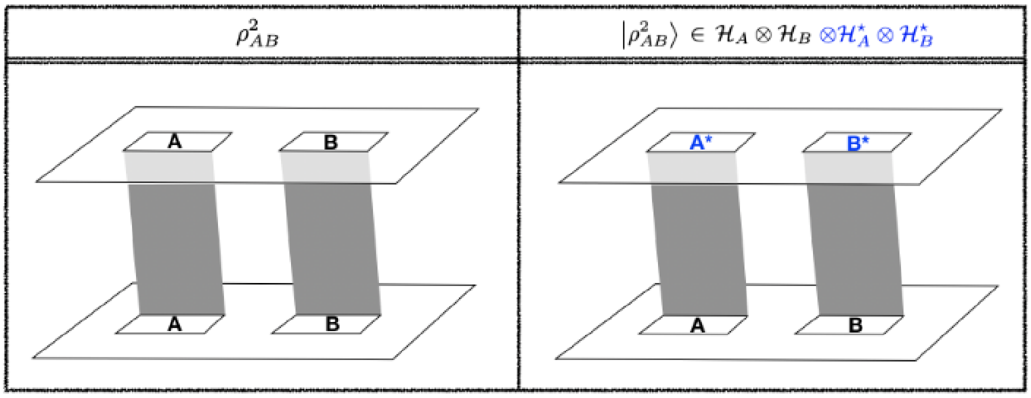


Figure 9: Picture credits [16]

Consider ρ^m_{AB}

$$\begin{aligned}\rho^m_{AB} &:= \sum_a p^m_a \left|\phi_a\right\rangle \left\langle\phi_a\right| \\ &= \sum_{a,i,j} p^m_a \sqrt{l^i_a l^j_a}\, |i_a\rangle_A |i_a\rangle_B \langle j_a|_A \langle j_a|_B\end{aligned}$$

This leads to a generalization of the canonically purified state which is as follows

$$\begin{aligned}\left|\rho^{m/2}_{AB}\right\rangle &:= \sum_{a,i,j} p^{m/2}_a \sqrt{l^i_a l^j_a}\, |i_a\rangle_A |i_a\rangle_B |j_a\rangle_{A^\star} |j_a\rangle_{B^\star} \\ |\psi_m\rangle &:= \frac{1}{\sqrt{\operatorname{Tr}\rho^m_{AB}}} \left|\rho^{m/2}_{AB}\right\rangle\end{aligned}$$

where $|\psi_m\rangle$ is introduced for proper normalization i.e

$$\operatorname{Tr}_{A^\star B^\star}\Big(|\psi_m\rangle\langle\psi_m|\Big) = \frac{\rho^m_{AB}}{\operatorname{Tr}\rho^m_{AB}}$$

The Renyi reflected entropy is defined as

$$S_n\left(AA^{\star}\right)_{\psi_m} := \frac{1}{1-n}\log \mathrm{Tr}_{AA^{*}}\left(\rho^{(m)}_{AA^{*}}\right)^n$$
$$\rho^{(m)}_{AA^{*}} := \mathrm{Tr}_{BB^{*}}\left|\psi_m\right\rangle\left\langle\psi_m\right|$$

This could be re-expressed interms of partition function $Z_{n,m}$ of mn sheeted Riemann surface which is as follow

$$S_R^{(m_e,n)}(AB) = S_n\left(AA^{\star}\right)_{\psi_{m_e}} = \frac{1}{1-n}\log\frac{Z_{n,m_e}}{\left(Z_{1,m_e}\right)^n}$$

where

$$Z_{n,m_e} := \mathrm{Tr}_{AA^{*}}\left(\mathrm{Tr}_{BB^{*}}\left|\rho_{AB}^{m_e/2}\right\rangle\left\langle\rho_{AB}^{m_e/2}\right|\right)^n$$

Note that$n\in\mathbb{Z}^{+}$and $m_e\in 2\mathbb{Z}^{+}$ such $m_e/2$ is a positive integer. In the limit $m=1$ and $n=1$ the above defined Renyi reflected entropy goes to the reflected entropy

$$S\left(AA^{\star}\right) = \lim_{n,m_e\to 1}\frac{1}{1-n}\log\frac{Z_{n,m}}{\left(Z_{1,m}\right)^n}$$

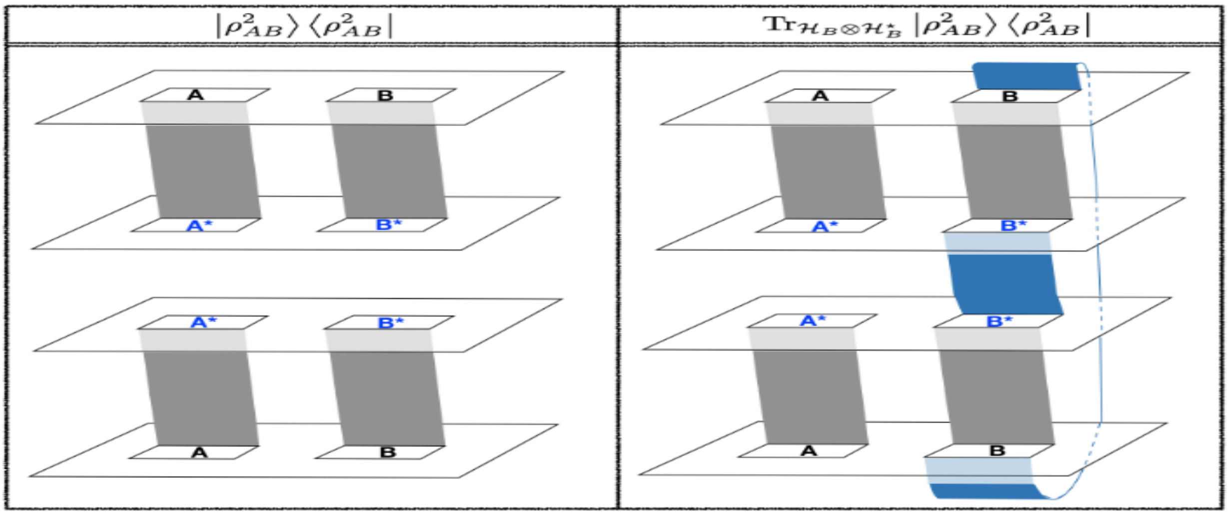


Figure 10: Picture credits [16]

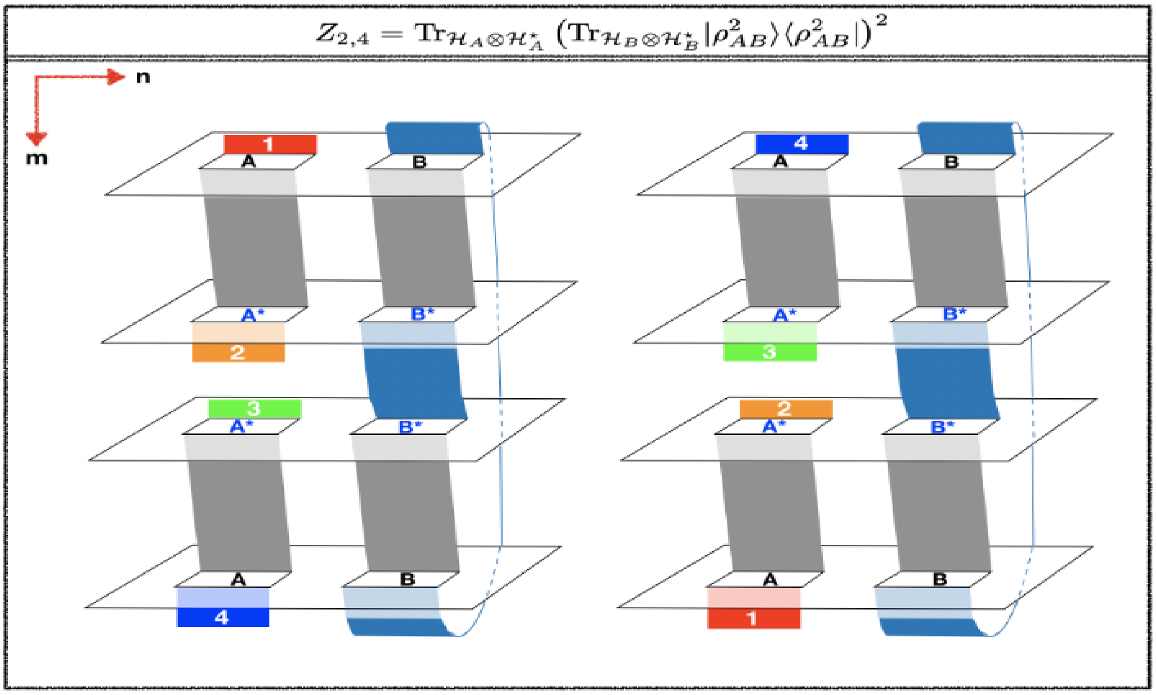


Figure 11: Picture credits [16]

For the case of two disjoint intervals

$$S_n(AA^\star)_{\psi_m} = \frac{1}{1-n} \log \frac{\langle \sigma_{g_A}(x_1)\sigma_{g_A^{-1}}(x_2)\sigma_{g_B}(x_3)\sigma_{g_B^{-1}}(x_4)\rangle_{CFT^{\otimes mn}}}{(\langle \sigma_{g_m}(x_1)\sigma_{g_m^{-1}}(x_2)\sigma_{g_m}(x_3)\sigma_{g_m^{-1}}(x_4)\rangle_{CFT^{\otimes m}})^n}$$

Twist operators are defined as follows

$$\sigma_{g_B} = \sigma_m^{\otimes n}, \quad \sigma_{g_B^{-1}} = \bar{\sigma}_m^{\otimes n}, \quad \sigma_{g_A} = \sigma_m'^{\otimes n}, \quad \sigma_{g_A^{-1}} = \bar{\sigma}_m'^{\otimes n}, \quad \sigma_{g_A^{-1} g_B} = \sigma_n^{(0)} \otimes \bar{\sigma}_n^{(m/2)}$$

The conformal block expansion for the four point correlation function is given by

$$\left\langle \sigma_{g_A}(x_1)\, \sigma_{g_A^{-1}}(x_2)\, \sigma_{g_B}(x_3)\, \sigma_{g_B^{-1}}(x_4) \right\rangle_{CFT^{\otimes mn}}$$
$$= \frac{1}{(x_4 - x_1)^{2(h+\bar{h})} (x_3 - x_2)^{2(h+\bar{h})}} \sum_p C_{ABp}^2 \mathcal{F}(mnc, h, h_p, 1-z)\, \mathcal{F}\left(mnc, \bar{h}, \bar{h}_p, 1-\bar{z}\right)$$

One may compute the dominant block in the limit

$$mnc \to \infty, \quad \epsilon := \frac{6h}{mnc} \quad and \quad \epsilon_p := \frac{6h_p}{mnc} \quad fixed$$

In the above limit and in the channel $z \to 1$ i.e when the intervals are close by

$$S_R(A:B) \sim \frac{c}{3} \log \left[\frac{1+\sqrt{1-z}}{1-\sqrt{1-z}}\right]$$

In the other channel $z \to 0$ the reflected entropy vanishes in the leading large central charge limit.

6 Holographic Entanglement Entropy

When Ryu-Takayanagi originally proposed their conjecture [10, 17], their motivation came from two different directions. Firstly, there were estimations which suggested that the entanglement entropy of a region A in a d-dimensional QFT has the coefficient of the leading divergent term proportional to the area of the boundary region-∂A that subsystem-A shares with the rest of the system

$$S_A = \gamma \frac{Area(\partial A)}{a^{d-1}} + .. \tag{6.1}$$

where a is the UV cut-off of the field theory with the understanding that the maximum contribution comes from the entanglement of modes that are near the boundary between A and A^c and therefore proprtional to area-A. The similarity between this and the Bekenstein-Hawking formula for the black hole entropy was one of the prime motivations

$$S_{BH} = \frac{\mathcal{A}}{4G_N^{d+1}}. \tag{6.2}$$

A stronger motivation comes from AdS_3/CFT_2 where one knows that the two point correlation function of the local operators in the dual CFT_{1+1} with large scaling dimensions ($O[c]$ in the large-c) is given by the exponential of the geodesic anchored to the points at which the operators are placed

$$\left\langle \mathcal{T}_n(u)\bar{\mathcal{T}}_n(v)\right\rangle \sim e^{-\Delta_n \mathcal{L}_{uv}} \tag{6.3}$$

which implies that the entanglement entropy of the CFT_{1+1} may be expressed in terms of the geodesic length in the corresponding dual AdS spacetime

$$S_A = \frac{\mathcal{L}_A}{4G_N^3} \tag{6.4}$$

6.1 Ryu-Takayanagi proposal

These motivations led to the celebrated Ryu-Takayanagi proposal which states that the entanglement entropy of the subsystem-A in a d-dimensional conformal field theory is given by the area of the static minimal surface-m_A which extends into the bulk AdS_{d+1} and is anchored to the subsystem-A

$$\boxed{S_A = \frac{Area(m_A)}{4G_N^{d+1}}} \tag{6.5}$$

In the presence of black holes $S_A \neq S_B$ as the dual CFT is in a thermal state, the geometric manifestation of this fact in terms of RT surfaces is follows. In the figure above one can see that if we use the RT conjecture as stated above one sees that

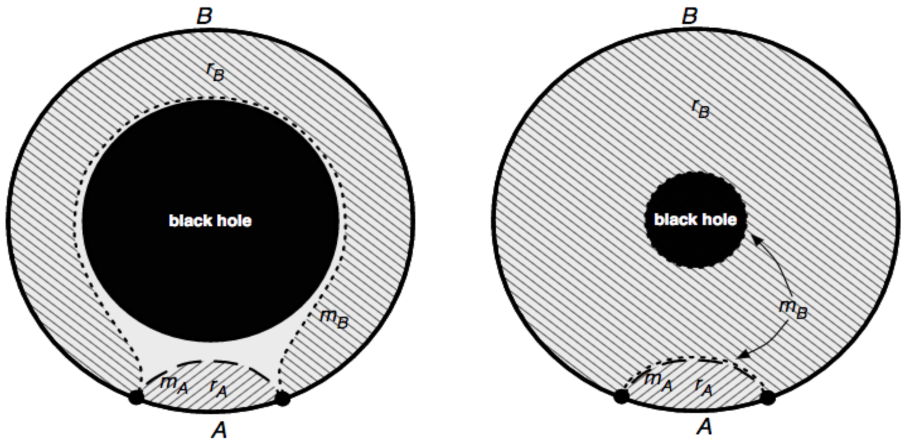


Figure 12: RT surfaces for the subsystems A and B in the presence of a blackhole in the bulk. Source: Headrick and Takayanagi 2007

there is an ambiguity i.e RT surface for the subsystem B can be chosen either as the one ending on A (as A and B share boundaries) or the one which goes around the blackhole. This is resolved by taking the RT surface to be homologous to the corresponding subsystem which essential means one has to find a region r in the bulk such that $\partial r = A \cup m_A$ (Similarly for computing S_B we have to find a region such that $\partial r = B \cup m_B$) . This discussion can be found in [18] Also this version of the conjecture is applicable to static spacetimes. A covariant generalization of the proposal was later proposed by Hubney, Rangamani and Takyanagi in [19]. Assuming just the AdS/CFT conjecture both of the RT and the HRT proposals have now been provem [20, 21].

6.1.1 Pure AdS_3

As a first check we will compute the entanglement entropy of the Vacuum of the CFT_{1+1} using RT formula for its bulk dual which is the pure AdS_3 spacetime and see if it matches with that obtained using the RT conjecture. The metric of the in Poincare coordinates is given by

$$ds^2 = \frac{R^2}{z^2}(-dt^2 + dx^2 + dz^2) \tag{6.6}$$

Where z is the inverse radial coordinate with boundary field theory living at $z = 0$. (x, t) are the coordinates on the boundary.Since the state is static we are allowed to take a constant time slice and the codimension-2 surface in AdS_3 geometry will be a space like geodesic which may be found by extremizing the following integral

$$\int ds = \int dz \frac{R}{z}\sqrt{x'(z)^2 + 1} \tag{6.7}$$

Note that since the boundary is at $z = 0$ the geodesic length diverges and needs to be regulated. This is the bulk manifestation of the fact that the entanglement entropy is UV divergent and is because of the famous UV-IR connection in AdS/CFT. Therefore the integral becomes

$$\mathcal{L}_A = 2\int_{z=\epsilon}^{z=z^*} dz \frac{R}{z}\sqrt{x'(z)^2+1} \tag{6.8}$$

where $z = z^*$ is the turning point of the geodesic. The Euler-Lagrange equation in this case is given by

$$\frac{R^2}{z^2}\frac{x'(z)}{\sqrt{x'(z)^2+1}} = const \tag{6.9}$$

At the turning point $z = z^*$, $\frac{dx}{dz} = 0$ this fixes the constant and we get

$$\frac{dx}{dz} = \frac{z}{(z^{*2}-z^2)^{\frac{1}{2}}} \tag{6.10}$$

Substituting this in eq.(6.8) we get the geodesic length to be

$$\mathcal{L}_A = 2\int_{z=\epsilon}^{z=z^*} dz \frac{R}{z}\frac{z^*}{(z^{*2}-z^2)^{\frac{1}{2}}} \tag{6.11}$$

Performing the integral and Substituting it in the expression for holographic entanglement entropy given by the RT conjecture

$$S_A = \frac{R}{2G_N}\log[\frac{\ell}{\epsilon}] \tag{6.12}$$

Identifying the ϵ as the UV cut off of the field theory and using the Brown-Hennaux formula $c = \frac{3R}{2G_N^{(3)}}$, one obtains th exact match with the CFT_{1+1} result

$$S_A = \frac{c}{3}\log[\frac{\ell}{a}] \tag{6.13}$$

6.1.2 Euclidean BTZ

According to AdS/CFT correspondence the finite temperature CFT_{1+1} is dual to the Euclidean BTZ black hole. We will compute the geodesic in this background and compare it with that of the CFT result discussed in the first lecture. The metric of the Euclidean BTZ black hole is given by

$$ds^2 = \frac{(r^2 - r_h^2)d}{R^2}\tau_E^2 + \frac{R^2}{(r^2 - r_h^2)}dr^2 + r^2 d\phi^2, \tag{6.14}$$

One may make the following coordinate transformation $r = r_h \cosh\rho, \tau_E = \frac{R}{r_h}\theta,\ \phi = \frac{R}{r_h}t$ to map this Euclidean AdS_3 in the global coordinates

$$ds^2 = R^2(d\rho^2 + \cosh^2\rho dt^2 + \sinh^2\rho d\theta^2). \tag{6.15}$$

This is done so as to evaluate the geodesic length utilizing the equivalence between Euclidean AdS_3 at temperature $T = \beta^{-1}$ and the Euclidean BTZ black hole at temperature $1/T$. The length of the RT geodesic is given by

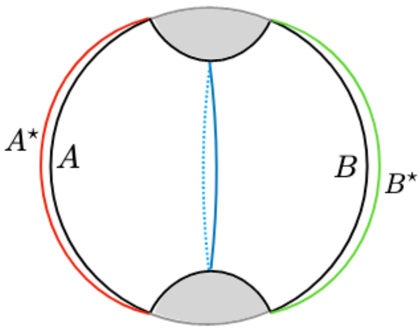
$$\mathcal{L}_A = 2R\ln\left[\frac{\beta_H}{\pi a}\sinh[\frac{\pi l}{\beta_H}]\right], \tag{6.16}$$

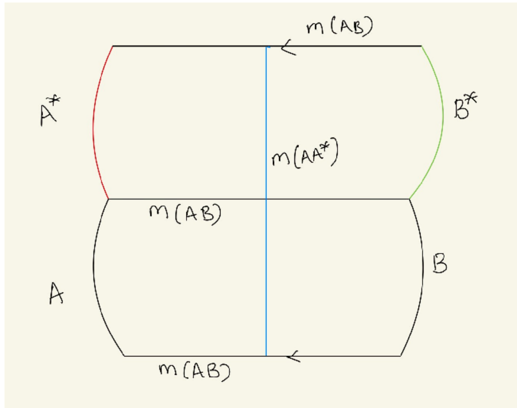
According the AdS/CFT dictionary the Hawking temperature of the black hole($T_H = \beta_H^{-1}$) is indentified with temperature of the CFT ($T = \beta^{-1}$). Therefore the entanglement entropy computed using the RT formula once again matches exactly with that of the CFT

$$S_A = \frac{c}{3}\ln\left[\frac{\beta}{\pi a}\sinh[\frac{\pi l}{\beta}]\right] \tag{6.17}$$

7 Reflected Entropy

Reasoning behind the holographic proposal





- Consider the entanglement wedge of the subsystem AB denoted as $r(AB)$ such that $\partial r(AB) = AB \cup m(AB)$ where $m(AB)$ is the Ryu-Takayanagi surface.
- Consider for example A and B as two disjoint regions in the dual CFT living in the assymptotic boundary of the bulk AdS space time. The density matrix ρ_{AB} corresponding to the region AB in general is a mixed state.
- The canonical purification of such a boundary region AB is equivalent to taking two copies of the entanglement wedges which are glued along the Ryu-Takayanagi surfaces such that the resulting manifold $rr^\star(AB) = r(AB) \cup r^\star(AB)$. This is the dual of the state $|\sqrt{\rho_{AB}}\rangle$. (This is a valid procedure to do. See [22]
- In this new manifold we can now define the entanglement wedge for the subsystem AA^* $r(AA^*)$ such that $\partial r(AA^*) = AA^* \cup m(AA^*)$.
- Consider splitting the RT surface $m(AB)$ into $m(AB) = \Gamma_A \cup \Gamma_B$ and finding a minimal surface $m\left(\Gamma_A A\right)$ inside $r(AB)$ that ends on $\partial\left(\Gamma_A A\right)$ and then

$$m\left(AA^\star\right) = m\left(\Gamma_A^{\min} A\right) \cup m\left(\Gamma_{A^\star}^{\min} A^\star\right) \tag{7.1}$$

where $\Gamma_A^{\min}$ is the minimal splitting surface. Hence

$$\boxed{S_R(A:B) = \frac{E_W(A:B)}{2G_N}} \tag{7.2}$$

7.1 Holographic Markov Gap

- The authors in [7] begin the holographic description of the Markov gap by considering the example of the two disjoint intervals in CFT_2 dual to Vacuum AdS_3.

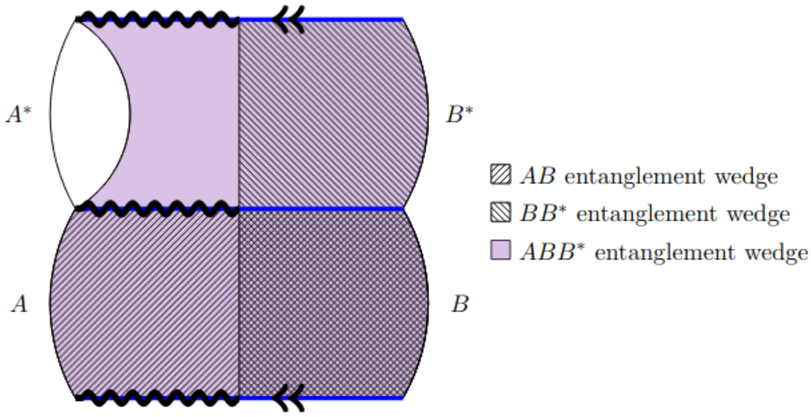


Figure 14: Picture credits [7]

- The black curvy lines around the RT surfaces (in blue) corresponding to the subsystem AB contain regions which are in the entanglement wedge of ABB^* but not in AB or BB^*.
- Another example involves the canonical purification of the two legs of a three boundary wormhole. Once again the black curvy lines around the RT surfaces (in blue) corresponding to the subsystem AB contain regions which are in the entanglement wedge of ABB^* but not in AB or BB^*.

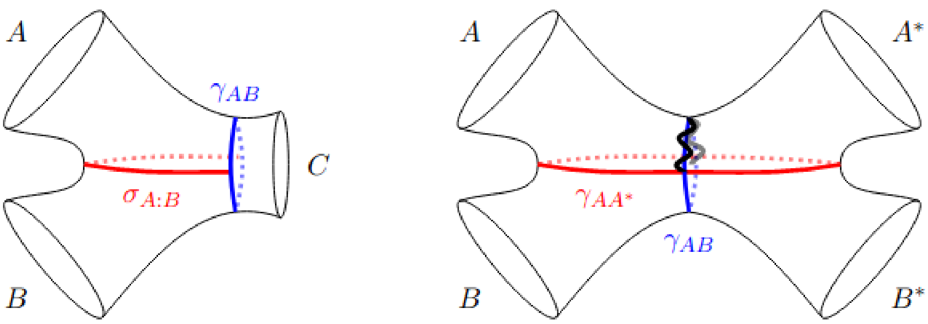


Figure 15: Picture credits [7]

- This implies that in holographic sysytems perfect quantum Markov recoveries are excluded and hence there exist non-trivial Markov gap.

7.2 Geometrizing the Markov gap

For any bipartite boundary state ρ_{AB} with a semiclassical dual, one can define a special surface homologous to A by taking the union of the entanglement wedge cross-section $E_W(A:B)$ together with the portions of the RT surface corresponding to AB that lie between $E_W(A:B)$ and A. The surface thus constructed is homologous to A. The authors call this surface the KRT surface KRT(A) — for "kinked Ryu-Takayanagi," because the surface will have right-angled kinks where $E_W(A:B)$ meets the RT surface of AB.

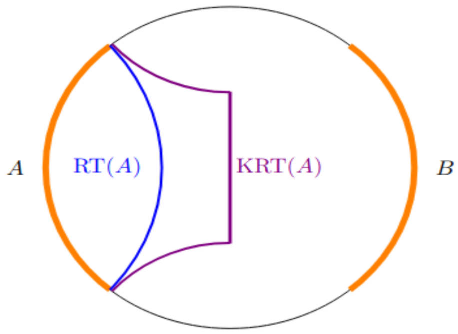


Figure 16: Picture credits [7]

$$S_R(A:B) - I(A:B) = \frac{2\,\text{area}(E_W(A:B)) - \text{area}(\text{RT}(A)) - \text{area}(\text{RT}(B)) + \text{area}(\text{RT}(AB))}{4G_N}$$
$$= \frac{\text{area}(\text{KRT}(A)) - \text{area}(\text{RT}(A))}{4G_N} + \frac{\text{area}(\text{KRT}(B)) - \text{area}(\text{RT}(B))}{4G_N}$$

where we have used

$$2\,\text{area}\,(E_W(A:B)) + \text{RT}(AB) = \text{area}(\text{KRT}(A)) + \text{area}(\text{KRT}(B))$$

The authors in [7] then utilizing some geometric relations in hyperbolic 2 manifolds to demonstrate that in AdS_3

$$\boxed{S_R(A:B) - I(A:B) \geq \frac{\log(2)\ell_{\text{AdS}}}{2G_N} \times (\#\text{ of cross-section boundaries }) + o\left(\frac{1}{G_N}\right)}$$

8 Entanglement Negativity

As discussed earlier, a replica technique proposed in [14, 15], was utilized to compute the entanglement negativity for various pure and mixed state configurations of a CFT_2. Following this, a holographic construction was advanced in [23–25] to determine the entanglement negativity of a holographic CFT_2 through a specific algebraic sum of the areas of extremal surfaces (lengths of geodesics in the dual bulk AdS_3). For example, the holographic entanglement negativity of two disjoint intervals A and B in proximity is given by [25]

$$\mathcal{E} = \frac{3}{16G_N}\left[\mathcal{L}_{A\cup C} + \mathcal{L}_{B\cup C} - \mathcal{L}_{A\cup B\cup C} - \mathcal{L}_C\right]. \tag{8.1}$$

$$= \frac{3}{4}\left[S(A\cup C) + S(B\cup C) - S(A\cup B\cup C) - S(C)\right] \tag{8.2}$$

where C denotes the interval sandwiched between A and B, $\mathcal{L}_Y$ denotes the length of a geodesic anchored on the subsystem Y, and G_N corresponds to the 3 dimensional gravitational constant. Note that in order to arrive at the last expression from the eq.(8.1), we have used the Ryu-Takayanagi proposal for holographic entanglement entropy which for a subsystem-Y is given as [10, 17, 19]

$$S_Y = \frac{\mathcal{L}_Y}{4G_N} \tag{8.3}$$

The numerical coefficient $\frac{3}{16G_N}$ in front of the area terms in eq.(8.1) has an important physical significance. In this context, it is crucial to recall that the holographic dual of the Renyi entropy of a subsystem-A in a CFT is given by the area of a cosmic brane with a tension in the dual bulk AdS spacetime [26]. This is expressed as follows

$$n^2\frac{\partial}{\partial n}\left(\frac{n-1}{n}S^{(n)}(A)\right) = \frac{\text{Area}(\text{ cosmic brane }_n)}{4G_N}$$
$$n^2\frac{\partial}{\partial n}\left(\frac{n-1}{n}\mathcal{A}^{(n)}\right) = \text{Area}(\text{ cosmic brane }_n) \tag{8.4}$$

where $S^{(n)}(A)$ is the n^{th} Renyi entanglement entropy for subsystem-A and the subscript n the RHS indicates that the tension of the cosmic brane depends on the replica index. Note that $\mathcal{A}^{(n)}$ is related to $S^{(n)}$ as follows

$$S^{(n)} = \frac{\mathcal{A}^{(n)}}{4G_N} \tag{8.5}$$

We will now utilize the following result which states that the quantity $\mathcal{A}^{(n)}$ related to the area of a back reacting cosmic brane is proportional to that of the corresponding cosmic brane with vanishing backreaction ($\mathcal{A}$) as described in [26–28]

$$\lim_{n\to 1/2}\mathcal{A}^{(n)} = \mathcal{X}_d^{hol}\mathcal{A}. \tag{8.6}$$

Observe that $\mathcal{X}_d$ in the above equation is a dimension dependent constant and the subscript d denotes the dimension of the holographic CFT_d. Note that the above relation holds only for configurations involving entangling surfaces with spherical symmetry and $\mathcal{X}_d$ is explicitly known to be of the following form

$$\mathcal{X}_d = \frac{1}{2} x_d^{d-2} \left(1 + x_d^2\right) - 1 \tag{8.7}$$

$$x_d = \frac{2}{d} \left(1 + \sqrt{1 - \frac{d}{2} + \frac{d^2}{4}}\right). \tag{8.8}$$

In the AdS_3/CFT_2 scenario this constant may be determined from the above expressions to be $\mathcal{X}_2 = \frac{3}{2}$. From the above discussion, it is clear that we may re-express the conjecture given in eq.(8.1) [25], as follows

$$\mathcal{E} = \frac{\mathcal{X}_2}{8G_N} \left[\mathcal{L}_{A\cup C} + \mathcal{L}_{B\cup C} - \mathcal{L}_{A\cup B\cup C} - \mathcal{L}_C\right]$$

We now utilize the result given in eq.(8.6), in the AdS_3/CFT_2 scenario i.e $\mathcal{L}^{(1/2)} = \chi_2 \mathcal{L}$, to rewrite the above expression as follows

$$\begin{aligned}
\mathcal{E} &= \frac{1}{8G_N} \left[\mathcal{L}^{(1/2)}_{A\cup C} + \mathcal{L}^{(1/2)}_{B\cup C} - \mathcal{L}^{(1/2)}_{A\cup B\cup C} - \mathcal{L}^{(1/2)}_C\right] && (8.9)\\
&= \frac{1}{2} \left[S^{(1/2)}(A\cup C) + S^{(1/2)}(B\cup C) - S^{(1/2)}(A\cup B\cup C) - S^{(1/2)}(C)\right], && (8.10)
\end{aligned}$$

where, $S^{(1/2)}(Y)$ in the above equation denotes the Renyi entropy of order half for the subsystem Y. In order to arrive at the last line of the above equation we have used eq.(8.5). Following the same procedure as above we may re-express the holographic conjecture for the entanglement negativity of the adjacent intervals in [24] as

$$\mathcal{E} = \frac{1}{2} \left[S^{(1/2)}(A) + S^{(1/2)}(B) - S^{(1/2)}(A\cup B)\right]. \tag{8.11}$$

Similarly, the holographic conjecture for the entanglement negativity of a single interval [23] may be expressed as follows

$$\mathcal{E} = \lim_{B_1\cup B_2 \to A^c} \frac{1}{2} \left[2S^{(1/2)}(A) + S^{(1/2)}(B_1) + S^{(1/2)}(B_2) - S^{(1/2)}(A\cup B_1) - S^{(1/2)}(A\cup B_2)\right] \tag{8.12}$$

Inspired by the above construction, we will propose below the island contribution to the entanglement negativity for various pure and mixed state configurations in terms of a combination of the generalized Renyi entropies of order half. However, before we discuss our island proposals, we briefly review the generalized Renyi entanglement entropy and comment on the analytic continuation to $n = \frac{1}{2}$.

Quite interestingly, when ABC is a tripartite system in pure state the disjoint, adjacent intervals and single interval all of the above proposals can be unified into one given by (see [29] for details)

$$\mathcal{E} = \frac{1}{2} I^{(1/2)}(A : B) \tag{8.13}$$

where $I^{(1/2)}$ is the mutual information of order half given as follows

$$I^{(1/2)}(A:B) = S^{(1/2)}(A) + S^{(1/2)}(B) - S^{(1/2)}(A \cup B). \tag{8.14}$$

For a certain class of holographic states namely the fixed area states, the above simplifies as these states have flat entanglement spectrum (which means any Renyi entropy is same as its entanglement entropy)

$$\mathcal{E} = \frac{1}{2} I(A:B) \tag{8.15}$$

For these fixed area recently it has been proven in [30] that the holographic entanglement negativity is indeed given by the above result. The proof interestingly involves a particular replica symmetry breaking saddle in the bulk.

There is an alternative proposal for entanglement negativity which says that for holographic states it is given by half of the Renyi reflected entropy of order half [31, 32]

$$\mathcal{E} = \frac{1}{2} S_R^{(1/2)}(A:B) \tag{8.16}$$

Since. we know that the reflected entropy is given by entanglement wedge cross section, the above equation reduces to

$$\mathcal{E} = EW^{(1/2)}(A:B) \tag{8.17}$$

$EW^{(1/2)}$ corresponds to the area of back reacted entanglement wedge cross section dual to the Renyi reflected entropy of order half.

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