Conformal field theory

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0.1 Introduction

Conformal transformations are defined as all those infinetesimal co-ordinate transformations which leave the metric invariant up to a scale.

$$g^{\mu\nu} - > \Lambda(x) g^{\mu\nu} \tag{1}$$

Using the transfomation rule for metric tensor

$$g^{'\mu\nu}(x) = \frac{\partial x^{'\mu}}{\partial x^{\rho}} \frac{\partial x^{'\nu}}{\partial x^{\sigma}} g^{\rho\sigma} \tag{2}$$

If we write the general transfomation as $x^{\mu} - > x^{\mu} + \epsilon^{\mu}(x)$ and substitue it in the above equation, and take up to first order in ϵ then we find the following relation

$$\partial^{\mu}\epsilon^{\nu} + \partial^{\nu}\epsilon^{\mu} = \frac{2}{d}(\partial.\epsilon) g^{\mu\nu} \tag{3}$$

In other words

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = \frac{2}{d}(\partial.\epsilon) g_{\mu\nu} \tag{4}$$

If we then modify this equation by taking derivative ∂^{ν} summing over ν we get

$$\partial_{\mu}\partial_{\nu}(\partial.\epsilon) + \Box\partial_{\nu}\epsilon_{\mu} = \frac{2}{d}\partial_{\mu}\partial_{\nu}(\partial.\epsilon) \tag{5}$$

Interchanging μ and ν and adding to the above equation one gets

$$(d-1)\Box(\partial.\epsilon) = 0 \tag{6}$$

For $d \neq 1$, the above equation says that for a conformal transfomation $\partial.\epsilon$ can at most be linear in x and hence, ϵ can atmost be quadratic in x.

0.2 CFT in d≥3

The above condition constrains the Conformal field transformations to be of 4 types for $d \geq 3$. $d = 2$ is a special case which will be discussed later. ϵ_{μ} can only be one of the four types below

$(1)\epsilon^{\mu} = a^{\mu} Translation$

$(2)\epsilon^{\mu} = \omega^{\mu}_{\nu} x^{\nu} Rotation$

$(3)\epsilon^{\mu} = \lambda x^{\mu} Dilations$

$(4)\epsilon^{\mu} = b^{\mu}x^2 - 2x^{\mu}(b.x)$

The last one is called the special conformal transfomation. Now, consider the variation in the form of the field under a coordinate transfomation of the above type then

$$\delta f(x') = f'(x') - f(x') \tag{7}$$

adding and subtracting f(x).

$\delta f(x') = f'(x') - f(x) + f(x) - f(x')$

$\delta f(x') = f'(x') - f(x) - \partial_{\mu} f(x)\delta x^{\mu}$

Comparing this with general equation for finding the generator of a symmetry transfomration for the variation in the functional form

$$\delta\Phi(x') = \Phi'(x') - \Phi(x) - i\omega_a T_a \Phi(x) \tag{8}$$

Whereω_a is the parameter and T_a the generator, we find the generators of the above mentioned conformal transformations as follows

$P_\mu = -i\partial_\mu$

D=-ix$^\mu \partial_\mu$

$M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$

$K_\mu = i(x^2 \partial_\mu - 2x_\mu x_\nu \partial^\nu)$

we can see that And the commutation relations can be easily found by plugging in the above expressions for each generator

$[D,K_\mu] = -iK_\mu$

$[D,P_\mu] = iP_\mu$

$[K_\mu, P_\nu] = 2i(\eta_{\mu\nu}D - M_{\mu\nu})$

$[K_\mu, M_{\nu\rho}] = 2i(\eta_{\mu\nu}K_\rho - \eta_{\mu\rho}K_\nu)$

$[P_\rho, M_{\mu\nu}] = i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu)$

$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\nu\rho}M_{\mu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho})$

We can clearly see from the above relations that the number of generators is finite and the algebra is closed and finite dimensional.

0.3 d=2

For two dimesional Eucledian space $g_{\mu\nu} = \delta_{\mu\nu}$(we can always go to minkowskian by making a wick rotation $x^0 -> ix^0$). Using equation(4) we get

$$\partial_0\epsilon_0 = \partial_1\epsilon_1$$
$$\partial_0\epsilon_1 = -\partial_1\epsilon_0$$

These resemble the Cauchy Riemann conditions that come up in complex analysis. So instead of using normal eucledian co-ordinates $x^0$and x^1, if we use complex coordinates $z = x^0 + ix^1$ and $\bar{z} = x^0 - ix^1$, then we can make use of complete machinary of complex analysis, but note that even though we use z and $\bar{z}$ as if they are two indpendent variables,at some point we will have to make the identifcation $\bar{z} = z^*$. However, unlike the higher dimensional case, we can now choose $\epsilon(z) = \epsilon^0 + i\epsilon^1$ to be any holomorphic function. So that $z -> f(z)$ where $f(z) = z + \epsilon(z)$ and $ds^2 = dzd\bar{z} -> (\frac{\partial f}{\partial z}\frac{\partial \bar{f}}{\partial \bar{z}})dzd\bar{z}$

0.4 Witt Algebra

As we have said above that $\epsilon(z)$ is holomorphic, (even meromorphic(containing isolated singularities) will do)we can write a Laurent expansion for it around $z = 0$ and we get

$$z' = z + \epsilon(z) = z + \sum c_n(-z^{n+1})(9)$$

$$\bar{z}- > \bar{z} + \bar{\epsilon}(\bar{z}) = \bar{z} + \sum \bar{c}_n(-\bar{z}^{n+1})(10)$$

where m and n are integers. Thus generators corresponding to these transfomrations are

$$l_n = -z^{n+1}\partial_z \quad (11)$$

$$\bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}} \quad (12)$$

The commutation relations therefore are given by

$$[l_m, l_n] = (m - n)l_{m+n} \quad (13)$$

$$[\bar{l_m}, \bar{l_n}] = (m - n)\bar{l_{m+n}} \quad (14)$$

Unlike SU(n) or the higher dimensional(d≥3) cases the algebra is not closed i.e any commutator between m and n goes to m+n th generator where m and n can be any integer . Therefore the algebra of CFTs in 2d spcae is infinite dimesional. This means that there are infinitely many conserved quantities and therefore any 2 dimensional conformally invariant theory is completely solvable.Note that if we demand that the generators to be globally defined i.e they are nonsingular in $S^2 = CU\infty$,then demanding non singular values of generator at 0 and ∞ constrains n $-1 \leq n \leq 1$. We say that $l_{-1}, l_0, l1$ generate globally defined conformal transfomation on Rieman sphere(S^2)

0.5 Virasaro Algebra

The Witt algebra of infinitesimal CFTs allows for what is called the central extension. A rough definiton of it is that, the central extension of a Witt algebra $\tilde{g} = g \oplus C$ which is charaterized by the commutation relations

$$[L_m, L_n] = (m - n)L_{m+n} + cp(m, n) \quad (15)$$

where $c \in C$ is called the central charge and p is bilinear $p : g * g \to C$ and a similar relations holds for commutation relations between $\bar{L_m}$and $\bar{L_n}$. Now using the Jacobi identity for $[[L_m, L_n], L_0]$ and its cyclic permutations we see that

$[[\mathrm{L}_m, L_n], L_0] + [[L_n, L_0], L_m] + [[L_0, L_m], L_n] = 0$

(m+n)p(m,n)=0

Which implies that for $m \neq -n$ we get $p(m, n) = 0$. Using the jacobi identity for $[[L_{-n+1}, L_n], L_{-1}]$ and its cyclic permutations, and normalizing $p(2, -2) = \frac{1}{2}$ we find that $p(n, -n) = \frac{1}{12}(n + 1)n(n - 1)$. Thus, we finally arrive at the commuation relations that charaterize Virasaro algebra

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \quad (16)$$

0.6 Applications:

Tracelessness of the energy-momentum tensor: An immideate application of the conformal invariance is this. We know form noether's theorem that any continuous symmetry gives rise to a conserved current given by

$$\jmath_\mu = T_{\mu\nu}\epsilon^\nu \tag{17}$$

Conservation implies

$$0 = \partial^\mu j_\mu = \partial^\mu T_{\mu\nu}\epsilon^\nu + T_{\mu\nu}\partial^\mu \epsilon^\nu \tag{18}$$

Using the energy-momentum conservation, the first term is zero. We now use the symmetric property of energy momentum tensor and using eq(4) we obtain

$$0 = 0 + \frac{1}{2}T_{\mu\nu}(\partial^\mu\epsilon^\nu + \partial^\nu\epsilon^\mu) = \frac{1}{2}T_{\mu\nu}\eta^{\mu\nu}(\partial.\epsilon)\frac{2}{d} = \frac{1}{d}T^\mu_\mu(\partial.\epsilon) \tag{19}$$

Which implies

$$T^\mu_\mu = 0 \tag{20}$$

Correlation fucntions: A very important application of Conformal invariance is that if we know how the field transforms, then we can directly get to the form of the correlation functions without even knowing the Lagrangian or the action(correlation fucntions are essential for deriving Feynman rules and hence for calculating any cross section after the interaction) We define here a theory with two types of fields: Primary fields and their subsets called quasi primary fields. we discuss correlations of only the quasiprimary fields which are subset of primary fields descirbed by the transfomation property

$$\phi_j(x) -> |\frac{\partial x'}{\partial x}|^{\frac{\Delta_j}{d}}\phi_j(x') \tag{21}$$

therefore the correlations transform as follows

$$< \phi_1(x_1)...\phi_n(x_n) >= |\frac{\partial x'}{\partial x}|^{\frac{\Delta_1}{d}}...|\frac{\partial x'}{\partial x}|^{\frac{\Delta_n}{d}} < \phi_1(x'_1)...\phi_n(x'_n) > \tag{22}$$

Rest of the fields can be expressed as linear combination of these fields. Let us look at the form of 2 point correlations of the above mentioned fields

$$< \phi_1(x_1)\phi_2(x_2) >= (|\frac{\partial x'}{\partial x}|^{\frac{\Delta_1}{d}})_{x=x_1}(|\frac{\partial x'}{\partial x}|^{\frac{\Delta_2}{d}})_{x=x_2} < \phi_1(x'_1)\phi_2(x'_2) > \tag{23}$$

Now we consider each of the four possible conformal transfomation and demand that the correlations do not change under those transfomrations. We can see translations and rotational transfomations are simple because the jacobian corresponding to them is unity. Translational invariance forces them to depend only on the differences between the coordinates and rotation forces them to depend only on the modulus. Therefore

$$< \phi_1(x_1)\phi_2(x_2) >= \frac{C_{12}}{r_{12}^a} \tag{24}$$

Where $r_{12} = |x_1 - x_2|$. Jacobian for dilations is $|\frac{\partial x'}{\partial x}| = \lambda^d$ where d is the dimension(space-time). Using the jacobian, eq(19) and (20) we find that $a = \Delta_1 + \Delta_2$.

$$< \phi_1(x_1)\phi_2(x_2) >= \frac{C_{12}}{r_{12}^{\Delta_1+\Delta_2}} \tag{25}$$

Finally using the special conformal transformation for which

$$|x_1' - x_2'|^2 = \frac{|x_1' - x_2'|^2}{(1+2b.x_1+b^2x_1^2)(1+2b.x_2+b^2x_2^2)} \tag{26}$$

and its jacobian $|\frac{\partial x'}{\partial x}| = \frac{1}{(1+2b.x+b^2x^2)^d}$ in eq(21) we find that correlation fucntions exist only when $\Delta_1 = \Delta_2 = \Delta$

$$< \phi_1(x_1)\phi_2(x_2) >= \frac{C_{12}}{r_{12}^{2\Delta}} \tag{27}$$

Exactly similar analysis can be done for 3 point correlations but 4 point onwards,correlations have to include cross ratios(which come because they are invariant under Special conformal transfomrations). Below are their forms

$$G^3(x_1,x_2,x_3) = \frac{C_{123}}{r_{12}^{\Delta_1+\Delta_2-\Delta_3} r_{23}^{\Delta_2+\Delta_3-\Delta_1} r_{31}^{\Delta_3+\Delta_1-\Delta_2}} \tag{28}$$

and

$$G^4(x_1,x_2,x_3,x_4) = F(\frac{r_{12}r_{34}}{r_{13}r_{24}}, \frac{r_{12}r_{34}}{r_{23}r_{14}}) \prod r_{ij}^{-(\Delta_i+\Delta_j)+\frac{\Delta}{3}} \tag{29}$$

d=2 Let us do the same in our 2d CFT. We define the primary fields which are the generalization of the metric transformation law $ds^2 = dzd\bar{z} -> (\frac{\partial f}{\partial z}\frac{\partial \bar{f}}{\partial \bar{z}})dzd\bar{z}$ as

$$\Phi(z,\bar{z}) -> (\frac{\partial f}{\partial z})^h \frac{\partial \bar{f}}{\partial \bar{z}})^{\bar{h}} \Phi(z', \bar{z}') \tag{30}$$

For the inifintesimal transfomations $z -> z + \epsilon(z)$ and $\bar{z} -> \bar{z} + \bar{\epsilon}(\bar{z})$, we see the variation in the field to be

$$\delta_{\epsilon,\bar{\epsilon}}\Phi(z,\bar{z}) = ((h\partial_z\epsilon + \epsilon\partial_z) + (\bar{h}\partial_{\bar{z}}\bar{\epsilon} + \bar{\epsilon}\partial_{\bar{z}})\Phi(z,\bar{z}) \tag{31}$$

For two point correlations to be invariant we should have

$$\delta_{\epsilon,\bar{\epsilon}}G^2(z_i,\bar{z}_i) =< \delta_{\epsilon,\bar{\epsilon}}\Phi_1, \Phi_2 > + < \Phi_1, \delta_{\epsilon,\bar{\epsilon}}\Phi_2 >= 0 \tag{32}$$

Now just as in the case of higher dimensions, we demand the invariance of $\epsilon(z) = 1$ and $\bar{\epsilon}(\bar{z}) = 1$ this forces G^2 depend only on $z_{12} = z_1 - z_2$and $\bar{z}_{12} = \bar{z}_1 - \bar{z}_2$. Similarly $\epsilon(z) = z, z^2$and $\bar{\epsilon}(\bar{z}) = \bar{z}, \bar{z}^2$ gives the correlation to be

$$G^2(z_i,\bar{z}_i) = \frac{c_{12}}{z_{12}^{2h}\bar{z}_{12}^{2\bar{h}}} \tag{33}$$

And for bosons which are defiend as those with conformal spin $s = h - \bar{h} = 0$ we get for $\Delta = h + \bar{h}$

$$G^2(z_i,\bar{z}_i) = \frac{c_{12}}{|z_{12}|^{2\Delta}} \tag{34}$$

Similar arguements can be used to obtain 3 point correlations as

$$G^2(z_i,\bar{z}_i) = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{13}^{h_3+h_1-h_2}} \frac{1}{\bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3} \bar{z}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1} \bar{z}_{13}^{\bar{h}_3+\bar{h}_1-\bar{h}_2}} \tag{35}$$

0.7 References

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